THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. IX. JANUARY, 1902. No. 1.

DETERMINATION OF ALL THE GROUPS OF ORDER 168.

By DR. G. A. MILLER, Leland Stanford Jr. University, Cal.

The first two composite numbers which are orders of simple groups are 60 and 168. Burnside determined the possible groups of the former of these orders in his Theory of Groups. The present paper is devoted to those of the latter order.

According to Sylow's theorem a group of order 168 contains either one or eight subgroups of order 7. If it contains 8 subgroups of order 7 it must permute them according to a transitive group of degree 8. As this transitive group involves substitutions of order 7 it must be doubly transitive. Hence such a group of order 168 must be isomorphic with a primitive group of degree 8 whose order is either 168 or 56. In the former case there is a simple isomorphism and the two possible groups are well known.* We proceed to prove that there is only one group of order 168 that contains 8 subgroups of order 7 and is isomorphic with the primitive group of degree 8 and order 56.

Since this primitive group contains an invariant subgroup of order 8 which involves 7 operators of order 2, such a group (G) contains an invariant subgroup (H) of order 24 which includes only operators of orders 2, 3, and 6. It contains only one subgroup of order 3. The two operators of order 3 must be invariant since the primitive group of degree 8 and order 56 does not include a subgroup of order 28. Hence H is composed of 7 operators of order 2 and 14 of order 6,

*Hoelder, *Mathematische Annalen*, Vol. 40, 1892, page 75; Cole, *Bulletin of the New York Mathematical Society*, Vol. 2, 1893, page 189.

besides the two operators of order 3 ; *i. e.* H is the direct product of its subgroups of orders 3 and 8. Moreover, G contains only one subgroup of order 24 since the group of degree 8 and order 56 contains only one subgroup of order 8.

In what follows we shall suppose that G is represented as a transitive group of degree 24.* Since H is regular and invariant G must be found in the holomorph of H. As the group of isomorphisms of H is the direct product of the groups of isomorphisms of its subgroups of orders 3 and 8,† all the subgroups of order 168 (including H) in the holomorph of H are conjugate ; *i. e. there is only one group of order* 168 *which contains* 8 *subgroups of order* 7 *and cannot be represented transitively on* 8 *letters.* Hence the total number of groups of order 168 which contain 8 subgroups of order 7 is three. In what follows we shall therefore assume that the groups under consideration contain only one subgroup of order 7.

1. Groups Containing Only One Subgroup of Order Eight.

If such a group contains only one subgroup of order 3, it is the direct product of its subgroups of orders 7, 8, and 3. Hence there are five such groups, corresponding to the five groups of order 8. If it contains four subgroups of order 3, its invariant subgroup of order 56 is the direct product of the group of order 7 and the quaternion group or the group of order 8 which involves no operator of order 4, since these are the only groups of order 8 whose groups of isomorphisms involve operators of order 3. The required groups must be in that part of the holomorphs of these groups of order 56 in which the operators of order 7 are invariant. Hence there are only two such groups, one for each of the given groups of order 8. They are the direct products of the two groups of order 24 which contain four subgroups of order 3 and only one subgroup of order 8.‡

When there are 7 subgroups of order 3 the group must be the direct product of the noncyclic group of order 21 and some subgroup of order 8, for each of its subgroups of order 24 contains only one subgroup of order 3. Hence, there are five such groups corresponding to the five groups of order 8. Finally, when there are 28 subgroups of order 3 the invariant subgroup of order 56 involves either the quaternion group or the group of order 7 which includes no operator of order 4. In each case there is just one group, since the operators of order 3 in the group of isomorphisms of these groups of order 8 are conjugate. Hence, there are just 14 groups of order 168 that contain only one subgroup of each of the orders 7 and 8.

2. Groups Containing Three Subgroups of Order Eight.

Each of these groups contains just three subgroups of order 56, which are the direct products of a subgroup of order 8 and the group of order 7. These conjugate subgroups of order 56 and the subgroups of order 8 are permuted

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*Dyck, *Mathematische Annalen*, Vol. 22, 1883, page 91; *Bulletin of the American Mathematical Society*, Vol. 3, 1896, page 215.

†*Transactions of the American Mathematical Society*, Vol. 1, 1900, page 396.

‡*Quarterly Journal of Mathematics*, Vol. 28, 1896, page 274.

according to the symmetric group of order 6. Hence each of the required groups contains invariantly the direct product of the group of order 7 and a group of order 4. If such a group has either one or four subgroups of order 3 it must be the direct product of the group of order 7 and a group of order 24 which has three subgroups of order 8 and either one or four subgroups of order 3. Hence there are just eight such groups, seven of them involve only one subgroup of order 3, while the remaining one involves four such subgroups.

There cannot be either 7 or 28 subgroups of order 3, since such a group would include 56 operators that are commutative with an operator of order 7 and therefore it could not contain just three conjugate subgroups of order 56 as was stated above.

3. Groups Containing Seven Subgroups of Order Eight.

The subgroups of order 8 are permuted according to the transitive group of degree 7 and order 14 when the number of subgroups of order 3 is either one or four. When the number of these subgroups is either 7 or 28 the subgroups of order 8 are permuted according to the group of order 42 and degree 7. Each of the groups under consideration contains an invariant subgroup of order 4. In the former case there must be an invariant subgroup of order 84, which is the direct product of the group of order 7 and one of the three groups of order 12 which include only one group of order 4.

Three of the required groups may be obtained by forming the direct products of the group of order 14 and degree 7 multiplied by one of the three groups of order 12 which have just been mentioned. When the given invariant subgroup of order 84 is cyclic there are three additional groups.* In one of these the subgroups of order 8 are cyclic, in the second they are the quaternion group, and in the third each of them contains just two operators of order 4. Hence, there are just four groups of order 168 that contain 7 subgroups of order 8 and also an operator of order 84. When the invariant subgroup of order 84 is noncyclic and abelian there are two additional groups, whose subgroups of order 8 are the abelian group which has two invariants and the group which can be represented on four letters, respectively. Finally, when the given subgroup of order 84 is nonabelian there is no group in addition to the direct product mentioned above. The total number of groups of order 168, which contain an invariant subgroup of order 12 and just seven subgroups of order 8 is therefore 8.

By forming the direct products of the metacyclic group of degree 7 and the groups of order 4, we obtain two groups containing seven subgroups of order three. Each of the groups which contain seven such subgroups includes the direct product of the semimetacyclic group of degree 7 and one of the groups of order 4. When this group of order 4 is cyclic there are three additional groups, whose subgroups of order 8 are distinct. When this group of order 4 is noncyclic there are two more groups. Hence there are just 7 groups containing 7 subgroups of order 3.

**Transactions of the American Mathematical Society*, Vol. 2, 1901, page 264.

When there are 28 subgroups of order 3, they are permuted according to a simply isomorphic group of order 168, which contains a subgroup of order 84. As the operators of order 2 in this subgroup cannot transform a subgroup of order 3 into itself the required groups (or group) must include operators of order 2 which are not in the subgroup of order 84. Hence there is only one such group, and the total number of groups of order 168 which contain seven subgroups of order 8 is 16.

4. Groups Containing Twenty-One Subgroups of Order Eight.

Such a group contains three conjugate subgroups of order 56, which it permutes according to the symmetric group of order 6. Each of these subgroups contains 7 groups of order 8, which it permutes according to a group of order 14. Hence it contains just one invariant subgroup of order 4, while the entire group G contains only one invariant subgroup of order 28. Since the quotient group of G with respect to this invariant subgroup is the symmetric group of degree 3, G cannot contain more than four subgroups of order 3; for, if there were more than four such subgroups, the operators of order 7 would be permuted according to the cyclic group of order 6 and hence this group would be a quotient group.

The invariant subgroup of order 28 includes either seven or just one subgroup of order 4. In the former case they are permuted under G according to the group of order 14, as there cannot be an invariant subgroup of order 4 in this case, and hence there must be an invariant subgroup of order 12. Since there is also such an invariant subgroup in the latter case, each of the groups under consideration contains an invariant subgroup of order 12, and hence also the direct product of the group of order 7 and some group of order 12.

When this direct product is cyclic, there are four such groups, which may be distinguished by their subgroups of order 8. When the direct product is the other abelian group of order 84, there are three additional groups. In two of these the subgroups of order 8 are abelian while those of the third can be represented on four letters. Each of these seven groups contains only one abelian group of order 84. It remains to determine the groups which do not include an abelian subgroup of order 84.

When the subgroup of order 12 is the alternating group there is only one G. Hence there is only one group of order 168 that includes 21 subgroups of order 8 and 4 subgroups of order 3. As each of the remaining groups contains three conjugate subgroups of order 4, its invariant subgroup of order 28 includes seven subgroups of order 4. When the subgroup of order 12 includes six operators of order 4, there are four additional groups; two of them contain just four operators of order 4 in each subgroup of order 8, while the non-abelian subgroups of order 8 in the other two are distinct. The first two of these four groups may be distinguished by the fact that only the first of them contains operators of order 12.

The only case which remains is when the subgroup of order 12 includes 7 operators of order 2. In this case there are again four groups. Two of these contain distinct abelian subgroups of order 8, while the other two include the

group of order 8 which can be represented on four letters. The latter may again be distinguished by the fact that only one of them contains operators of order 12. This concludes the examination of all the possible cases and proves that the total number of groups of order 168 that have 21 subgroups of order 8 is the same as the number of those containing 7 such subgroups, viz. 16. *The total number of groups of order* 168 *is therefore* 57. Only one of these is insolvable.

ON LIMITS.

By DR. ARNOLD EMCH, University of Colorado.

In most of the elementary text-books of algebra and calculus the chapters on limits are treated in a rather indefinite manner. I suspect that the reason for this deficiency lies partly in the semi-philosophical nature of the subject, partly in the neglect of the authors to apply the results of the theory of functions to limitting processes.

As an example I mention the frequently occurring definition of the *limit of a variable.*

(a) "When according to its law of change, a variable approaches indefinitely near a constant, but can never reach it, the constant is called the limit of the variable."

This definition restricts the variable to the members of a sequence as it will appear from the following proposition given by Harkness and Morley :*

I. "The numbers ξ_1, ξ_2, ξ_3, ... of a sequence are said to tend to the limit a, when to every positive number ε there corresponds a positive integer μ such that for ξ_μ and for all later members ξ_n of the sequence we have

$$| \xi_n - a | < \varepsilon."$$

a itself does not belong to the sequence. Terms in the sequence of all positivs integers 1, 2, 3, ...—, ∞ does not belong to the sequence, although ∞ is the limit of the sequence. Similarly, in the sequence

$$\tfrac{1}{1}, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots .$$

0, which is its limit, does not belong to the sequence. Notice also that 0 and ∞, which shall be defined presently, are among the values which a limit can have.

An infinitely large quantity, or the symbol ∞ is defined as the indefinite quantity A which satisfies the inequality

**Introduction to Analytic Functions*, page 67.
See also Burkhardt: *Functionentheorie*, Vol. I., page 68.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

150. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

A commission merchant sold $\$W,=\$4750$ worth of wheat. After deducting his commission at $m\%,=3\%$, purchased with the proceeds a draft at $d,=60$ days at $r\%,=10\%$, interest, and at $p\%,=\frac{3}{4}\%$ premium. What was the face of the draft ?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and P. S. BERG, B. Sc., Principal of Schools, Larimore, N. D.

$$\$W(100\%-m\%)=\$4750\times.97=\$4607.50.$$

$$(d+3)\text{ days}=63\text{ days}, \frac{(d+3)r\%}{36}=\frac{63\times10\%}{36}=.0175.$$

$$100\%+p\%-\frac{(d+3)r\%}{36}=1.0075-.0175=.99.$$

$$\$6407.50\times.99=\$6343.425.$$

151. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

A merchant marked a lot of goods $m\%=20\%$, above cost; but, in consequence of a rise in the market price, he marked up the goods $n\%=10\%$, on the marked price. What per cent. was the last selling price of the goods ? What would be his gain on sales amounting to $\$S=\$5780.50$?

Solution by JAMES F. LAWRENCE, A. B., Instructor in Mathematics, Rogers Academy, Rogers, Ark.

1. Let $r\%=100\%=$cost.
2. $\therefore$ $r\%+m\%=$marked price.
3. Let $p\%=100\%=$marked price$=r\%+m\%$.
4. $1\%=\frac{r+m}{p}\%$.
5. $(p+n)\%=\frac{(r+m)(p+n)}{p}\%=132\%=$selling price.
6. $\therefore$ $32\%=$gain on cost price.
7. But $\frac{(r+m)(p+n)}{p}\%=\$S=\$5780\frac{1}{2}$.
8. $\therefore$ $r=100\%=\frac{rpS}{(r+m)(p+n)}=\$4379\frac{1}{6}=$cost.
9. $\therefore$ $\$1401\frac{1}{3}=\$5780\frac{1}{2}-\$4379\frac{1}{6}=$gain.

Also solved by *T. T. DAVIS*, Principal of High School, Portland Oregon; *G. B. M. ZERR*, *C. A. LINDEMAN*, and *P. S. BERG*.

ALGEBRA.

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129. Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, O.

$$\left|\begin{array}{lllll} 1-\binom{m-1}{0} & -\binom{m-1}{1} & \cdot\cdot & & \binom{m-1}{m-2} \\ 1 & 1 & -\binom{m-2}{0} & \cdot\cdot & \binom{m-2}{m-3} \\ 1 & 0 & 1 & \cdot\cdot & \binom{m-3}{m-4} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & 1 & \binom{1}{0} \\ 1 & 0 & 0 \ldots\ldots & 0 & 1 \end{array}\right| e=\sum_{n=1}^{n=\infty}\frac{n^m}{n\,!}. \text{ where } \binom{m-2}{k}=\frac{(n-2)\,..\,(m-k-1)}{k\,!}$$

Solution by the PROPOSER.

$$\text{Let } f(x)=e^{e^x}=1+e^x+\frac{e^{2x}}{2\,!}+\frac{e^{3x}}{3\,!}+\frac{e^{4x}}{4\,!}+\frac{e^{5x}}{5\,!}+\,..\,+\frac{e^{rx}}{r\,!}+\ldots.$$

$$=1+(1+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\frac{x^4}{4\,!}+\frac{x^5}{5\,!}+\ldots.)$$

$$+\frac{1}{2\,!}(1+2x+\frac{(2x)^2}{2\,!}+\frac{(2x)^3}{3\,!}+\frac{(2x)^4}{4\,!}+\frac{(2x)^5}{5\,!}+\ldots.)$$

$$+\frac{1}{3\,!}(1+3x+\frac{(3x)^2}{2\,!}+\frac{(3x)^3}{3\,!}+\ \ldots)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{1}{r\,!}(1+rx+\frac{(rx)^2}{2\,!}+\frac{(rx)^3}{3\,!}+\ldots.)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$=e+\sum_{r=1}^{r=\infty}a_rx^r, \text{ where } a_r=\frac{1}{r\,!}\sum_{k=1}^{k=\infty}\frac{k^r}{k\,!}.$$

$$\text{But } f(x)=f(0)+\frac{f'(0)}{1\,!}x+\frac{f''(0)}{2\,!}x^2+\ldots.$$

$$\therefore f^m(0)=\sum_{k=1}^{k=\infty}\frac{k^m}{k\,!}\ldots.(1).$$

Let $y=e^{e^x}$, then $\log y=e^x$.

$\therefore \dfrac{dy}{dx} = ye^x$. And by Leibnitz's Formula,

$$f^m(0) = f^{m-1}(0) + (m-1)f^{m-2}(0) + \frac{(m-1)(m-2)}{2\,!} f^{m-3}(0) + \dots\dots$$

$$+ (m-1)f'(0) + f(0) = \sum_{k=1}^{k=\infty} \frac{k^m}{k\,!}. \text{ [by (1)]} \quad \dots (2).$$

Or, letting x_m represent $f^m(0)$, etc.,

$$x_m - x_{m-1} - C_{m-1,1}x_{m-2} - \dots\dots - C_{m-1,1}x_1 = f(0) = e.$$

$$x_{m-1} - x_{m-2} - C_{m-2,1}x_{m-3} \dots\dots - C_{m-2,1}x_1 = e$$

. . . .

. . . .

$$x_2 - x_1 = e$$

$$x_1 = e$$

$$\therefore x_m = \begin{vmatrix} e & -\binom{m-1}{0} & -\binom{m-1}{1} & & \dots\dots\dots & -\binom{m-1}{m-2} \\ e & 1 & -\binom{m-2}{0} & & \dots\dots\dots & -\binom{m-2}{m-3} \\ e & 0 & 1 & & \dots\dots\dots & -\binom{m-3}{m-4} \\ e & 0 & 0 & 1 & \dots\dots & -\binom{m-4}{m-5} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ e & 0 & 0 & 0 & \dots\ 1 & -\binom{1}{0} \\ e & 0 & 0 & 0 & \dots\ 0 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & -\binom{m-1}{0} & -\binom{m-1}{1} & & \dots\dots\dots & -\binom{m-1}{m-2} \\ 1 & 1 & -\binom{m-2}{0} & & \dots\dots\dots & -\binom{m-2}{m-3} \\ 1 & 0 & 1 & & \dots\dots\dots & -\binom{m-3}{m-4} \\ 1 & 0 & 0 & 1 & \dots\dots & -\binom{m-4}{m-5} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 & \dots\dots 1 & -\binom{1}{0} \\ 1 & 0 & 0 & 0 & \dots\dots 0 & 1 \end{vmatrix} e = \sum_{k=1}^{k=\infty} \frac{k^m}{k\,!} \dots\dots (3).$$

By using (2) or (3) the following special relations have been obtained :

$$2e=\sum_{k=1}^{k=\infty}\frac{k^2}{k!},\ 5e=\sum_{k=1}^{k=\infty}\frac{k^3}{k!},\ 15e=\sum_{k=1}^{k=\infty}\frac{k^4}{k!},\ 52e=\sum_{k=1}^{k=\infty}\frac{k^5}{k!},\ 203e=\sum_{k=1}^{k=\infty}\frac{k^6}{k!},\ 877e=\sum_{k=1}^{k=\infty}\frac{k^7}{k!},$$

$$4140e=\sum_{k=1}^{k=\infty}\frac{k^8}{k!},\ 21147e=\sum_{k=1}^{k=\infty}\frac{k^9}{k!},\ 115975e=\sum_{k=1}^{k=\infty}\frac{k^{10}}{k!}.$$

Also solved by *G. B. M. ZERR.*

GEOMETRY.

160. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

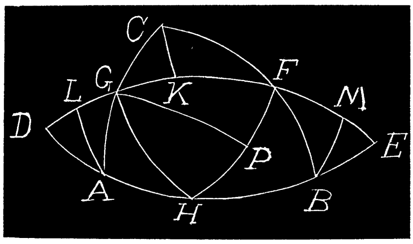
Let GFH be the spherical triangle formed by joining the mid-points of the sides of the spherical triangle ABC; E the spherical excess of ABC; β, p the base and altitude of GFH. Prove $\sin\frac{1}{2}E=\sin\beta\sin p$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let $BA=c$, $GF=\gamma$, $FH=\beta$, $GH=\delta$, $GP=p$.

Draw AL, BM, CK perpendicular to $DGFC$. Now $DABE=DGFE=\pi$.

$AD=BE=\frac{1}{2}(\pi-c)$, $DL=ME$, $LG=CK$, $KF=FM$.



$\therefore 2(DL+GK+KF)=\pi$. Also $2GK+2KF=2\gamma$.

$\therefore 2DL+2\gamma=\pi$, or $DL=\frac{1}{2}\pi-\gamma$.

$\angle DAL=\angle EBM$, $\angle LAG=\angle GCK$, $\angle KCF=\angle FBM$.

$\therefore 2\angle DAL+C+A+B=2\pi$.

$\therefore \angle DAL=\pi-\frac{1}{2}(A+B+C)=\pi-s=\frac{1}{2}(\pi-E)$.

$\cos DAL=\sin D\cos DL$. $\cos\frac{1}{2}(\pi-E)=\sin D\cos(\frac{1}{2}\pi-\gamma)$.

$\therefore \sin\frac{1}{2}E=\sin D\sin\gamma$.

Now $\sin D : \sin DFH=\sin\beta : \sin DH$. But $DH=\frac{1}{2}\pi$.

$\therefore \sin D=\sin\beta\sin DFH$. But $\sin p=\sin\gamma\sin DFH$.

$\therefore \sin D=(\sin\beta\sin p)/\sin\gamma$.

$\therefore \sin\frac{1}{2}E=\sin\beta\sin p$.

Also solved by *J. SCHEFFER* and *L. C. WALKER.*

161. Proposed by MARCUS BAKER, U. S. Coast and Geodetic Survey Office, Washington, D. C.

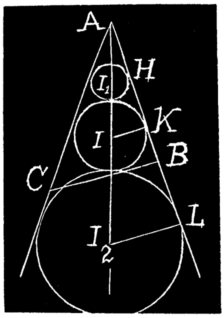
A circle, radius r, is inscribed in a triangle ABC. In the angles A, B, and C are inscribed circles each touching two sides and the inscribed circle. There are six such circles. The first group of three have their centers between the in-centers and the vertices, and the second group of three does not. Let r_a, r_b, r_c denote the radii of the first group. Then this well known relation holds: $r=\sqrt{(r_ar_b)}+\sqrt{(r_br_c)}+\sqrt{(r_cr_a)}$. Let R_a, R_b, R_c denote the radii of the second group. Then this relation holds:

$$\frac{1}{r}=\frac{1}{\surd(R_aR_b)}+\frac{1}{\surd(R_bR_c)}+\frac{1}{\surd(R_cR_a)}.$$

Required proof.

I. Solution by LON C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Let ABC be a triangle, I the center of the in-circle with radius r. The circles with radii r_a, R_a will have centers I_1, I_2, and points of contact with the in-circle on the straight line AI. Draw the radii I_1H, IK, I_2L. The length of IA is



$r+r_a+r_a\text{cosec}\frac{1}{2}A=r\text{cosec}\frac{1}{2}A$; thence

$$r_a=\frac{r(1-\sin\frac{1}{2}A)}{1+\sin\frac{1}{2}A}=\frac{r(\cos\frac{1}{4}A-\sin\frac{1}{4}A)^2}{(\cos\frac{1}{4}A+\sin\frac{1}{4}A)^2},$$

and $$\surd(r_ar_b)=\frac{r(\cos\frac{1}{4}A-\sin\frac{1}{4}A)(\cos\frac{1}{4}B-\sin\frac{1}{4}B)}{(\cos\frac{1}{4}A+\sin\frac{1}{4}A)(\cos\frac{1}{4}B+\sin\frac{1}{4}B)}$$

$$=\frac{r\cos\frac{A+\pi}{4}\cos\frac{B+\pi}{4}\cos\frac{C+\pi}{4}}{\cos\frac{A-\pi}{4}\cos\frac{B-\pi}{4}\cos\frac{C-\pi}{4}}=\frac{r(\cos\frac{1}{2}A+\cos\frac{1}{2}B-\cos\frac{1}{2}C)}{\cos\frac{1}{2}A+\cos\frac{1}{2}B+\cos\frac{1}{2}C}.$$

In like manner can be found $\surd(r_br_c)$ and $\surd(r_cr_a)$. Then by adding,

$$r=\surd(r_ar_b)+\surd(r_br_c)+\surd(r_cr_a).$$

The length of I_2A is $R_a+r+\text{cosec}\frac{1}{2}A=R_a\text{cosec}\frac{1}{2}A$; thence

$$\frac{1}{R_a}=\frac{1-\sin\frac{1}{2}A}{r(1+\sin\frac{1}{2}A)},\text{ as before,}$$

$$\frac{1}{\surd(R_aR_b)}=\frac{\cos\frac{1}{2}A+\cos\frac{1}{2}B-\cos\frac{1}{2}C}{r(\cos\frac{1}{2}A+\cos\frac{1}{2}B+\cos\frac{1}{2}C)}.$$

Similarly, can be found $\dfrac{1}{\surd(R_bR_c)}$ and $\dfrac{1}{\surd(R_cR_a)}$. Then by adding,

$$\frac{1}{r}=\frac{1}{\surd(R_aR_b)}+\frac{1}{\surd(B_bR_c)}+\frac{1}{\surd(R_cR_a)}.$$

II. Solution by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud. Gloucestershire, England.

It is clear that $r\text{cosec}\frac{1}{2}A+r+R_a=R_a\text{cosec}\frac{1}{2}A$.

$$\therefore \Sigma\frac{1}{\surd(R_aR_b)}=\frac{1}{r}\Sigma\sqrt{\frac{(1-\sin\frac{1}{2}A)(1-\sin\frac{1}{2}B)}{(1+\sin\frac{1}{2}A)(1+\sin\frac{1}{2}B)}}$$

$$=\frac{1}{r}\frac{\Sigma(\cos\frac{1}{4}A-\sin\frac{1}{4}A)(\cos\frac{1}{4}B-\sin\frac{1}{4}B)(\cos\frac{1}{4}C+\sin\frac{1}{4}C)}{\pi(\cos\frac{1}{4}A+\sin\frac{1}{4}A)(\cos\frac{1}{4}B+\sin\frac{1}{4}B)(\cos\frac{1}{4}C+\sin\frac{1}{4}C)}$$

$$=\frac{1}{r}\frac{\Sigma\cos\dfrac{A+\pi}{4}\cos\dfrac{B+\pi}{4}\cos\dfrac{C-\pi}{4}}{\pi\cos\dfrac{A-\pi}{4}}$$

$$=\frac{1}{r}\frac{\Sigma(\cos\frac{1}{2}A+\cos\frac{1}{2}B-\cos\frac{1}{2}C)}{\Sigma\cos\frac{1}{2}A}=\frac{1}{r}.$$

Also solved by *H. C. WHITAKER*, and *G. B. M. ZERR*.

162. Proposed by J. D. PALMER, Providence, Ky.

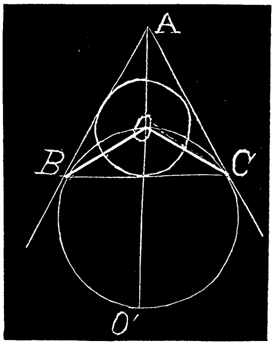
Given the distances from the vertices of a triangle, ABC, to the center of the circle, to construct the triangle.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

AO, BO, $CO=a$, b, c, respectively, where O is the center of the in-circle; BC, AC, $AB=x$, y, z, respectively. Let O_1 be the center of the ex-circle opposite A. Then

$$AO^2=\frac{(p-x)^2}{\cos^2\frac{1}{2}A},\ \text{where}\ p=\tfrac{1}{2}(x+y+z).$$

$$\therefore AO^2=\left[\frac{p-x}{p}\right]yz=yz-\frac{xyz}{p}=yz-4Rr=a^2.$$



$$\text{Similarly, } BO^2=\left[\frac{p-y}{p}\right]xz=xz-4Rr=b^2.$$

$$CO^2=\left[\frac{p-z}{p}\right]xy=xy-4Rr=c^2.$$

$$\therefore AO.BO.CO=\frac{\triangle\, xyz}{p^2}=4R^2r^2 \text{ or } 2\surd(abc)=4Rr$$

$$\therefore a^2+2\surd(abc)=yz,\ b^2+2\surd(abc)=xz,$$

$$c^2=2\surd(abc)=xy.$$

$$\therefore x[a^2+2\surd(abc)=y[b^2+2\surd(abc)]=z[c^2+2\surd(abc)]$$

$$=\sqrt{\{[a^2+2\sqrt{(abc)}][b^2+2\sqrt{(abc)}][c^2+2\sqrt{(abc)}]\}}.$$

This gives us the values of the sides.

Otherwise draw AO and produce AO to O_1 so that $OO_1=2\sqrt{(bc/a)}$. Upon OO_1 as diameter describe a circle. With O as a center and b as a radius describe an arc cutting the circle in B. Similarly, with O as center and c as radius, draw an arc cutting the circle in C. Join BC, AC, AB, then ABC is the triangle required. For O_1 is the ex-center opposite A by construction as follows:

$AO_1=p/\cos\frac{1}{2}A$. $\therefore$ $AOAO_1=yz=AO^2+2\sqrt{(AO.BO.CO)}$.

$\therefore$ $AO_1=AO+2\sqrt{[(BO.CO)/AO]}$.

CALCULUS.

121. **Proposed by W. W. LANDIS, A. M., Professor of Mathematics and Astronomy, Dickinson College, Carlisle, Pa.**

Solve the differential equation $\left[\frac{d}{dx}+b\right]^n y=\cos ax$.

Solution by LON C. WALKER, A. M., Petaluma High School, Petaluma, Cal., and LEWIS NEIKIRK, B. S., Boulder, Col.

$$\left[\frac{d}{dx}+b\right]^n y=\cos ax.$$

$\left[\frac{d}{dx}+b\right]^n$ has n roots each $=-b$.

$\therefore$ Comp. Factor$=e^{-bx}(c_1+c_2x+c_3x^2+c_4x^3+\ldots.c_nx^{n-1})$.

$$\frac{1}{\left[\frac{d}{dx}+b\right]^n}\cos ax=\frac{\left[\frac{d}{dx}-b\right]^n}{\left[\frac{d}{dx}-b^2\right]^n}\cos ax=\left\{\frac{b-\frac{d}{dx}}{a^2+b^2}\right\}^n\cos ax.$$

Let $n=1$. $\therefore$ $\frac{b-d/dx}{a^2+b^2}\cos ax=\frac{1}{a^2+b^2}(b\cos ax+a\sin ax)$

$$=\frac{1}{(a^2+b^2)^{\frac{1}{2}}}\left[\frac{b\cos ax+a\sin ax}{\sqrt{(a^2+b^2)}}\right]\ldots.(1).$$

Put $\theta=\cot^{-1}b/a$, then $\sin\theta=\frac{a}{\sqrt{(a^2+b^2)}}$, and $\cos\theta=\frac{b}{\sqrt{(a^2+b^2)}}$.

$\therefore$ (1) reduces to

$$\frac{b-d/dx}{a^2+b^2}\cos ax=(a^2+b^2)^{-\frac{1}{2}}(\cos\theta\cos ax+\sin\theta\sin ax)=(a^2+b^2)^{-\frac{1}{2}}\cos(ax-1\theta).$$

When $n=2$, $\dfrac{b-d/dx}{a^2+b^2}\left[\dfrac{1}{(a^2+b^2)^{\frac{1}{2}}}\cos(ax-\theta)\right]$

$$=\frac{1}{a^2+b^2}\left[\frac{b\cos(ax-\theta)+a}{\sqrt{(a^2+b^2)}}\sin(ax+\theta)\right]$$

$$=\frac{1}{a^2+b^2}\left[\cos\theta\cos(ax-\theta)+\sin\theta\sin(ax-\theta)\right]=(a^2+b^2)^{-\frac{2}{2}}\cos(ax-2\theta).$$

When $n=3$, $\therefore$ $\dfrac{b-d/dx}{a^2+b^2}\left[\dfrac{1}{(a^2+b^2)^{\frac{1}{2}}}\cos(ax-\theta)\right]$

$$=\frac{1}{a^2+b^2}\left[\frac{b\cos(ax-2\theta)a\sin(ax-2\theta)}{\sqrt{(a^2+b^2)}}\right]=(a^2+b^2)^{-\frac{3}{2}}\cos(ax-3\theta).$$

This method holds for $(n-1)$ terms.

Hence $y=e^{-bx}(c_1+c_2x+c_3x^2+c_4x^3+\ldots.c_nx^{n-1})$

$$+(a^2+b^2)^{-\frac{1}{2}n}\cos[ax-n\cot^{-1}(b/a)].$$

SECOND SOLUTION.

Put $D=\dfrac{d}{dx}$, then we have $\dfrac{1}{(D+b)^n}\cos ax=\dfrac{(D-b)^n}{(D^2-b^2)^n}\cos ax$

$$=\frac{(-1)^n(b-D)^n}{(a^2+b^2)^n}\cos ax.$$

The numerator $=(-1)^n\left[b^n\cos ax+nab^{-1}\sin ax)-\dfrac{n(n-1)}{2!}a^2b^{n-2}\cos ax\ldots\right]$

$$=(-1)^n\cos[ax-n\cot^{-1}(b/a)]$$

The value of y is the same as in I.

Also solved by *WILLIAM HOOVER*, *G. B. M. ZERR*, and the *PROPOSER*.

Professor Landis remarks that the exponent of a^2+b^2 is given $\frac{1}{2}n$ in Johnson's *Differential Equations* (problem 17, page 122) instead of the correct result $-\frac{1}{2}n$.

122. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

Solve the differential equation $(y-x)\sqrt{(1+x^2)}\dfrac{dy}{dx}=n(1+y^2)^{\frac{3}{2}}$

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

Let $x=\tan\theta$, $y=\tan\phi$, then $\sin(\phi-\theta)d\phi=nd\theta\ldots.(1)$.

$$(\tan\varphi-\tan\theta)\sqrt{(1+\tan^2\theta)}\frac{\sec^2\varphi d\varphi}{\sec^2\theta d\theta}=n(1+\tan^2\varphi)^{\frac{3}{2}}=n\sec^3\phi\ldots.(2).$$

$$(\tan\varphi-\tan\theta)d\phi=n\sec\phi\sec\theta d\theta\ldots.(3).$$

$$\frac{\sin\varphi\cos\theta-\cos\varphi\sin\theta}{\cos\varphi\cos\theta}d\varphi=\frac{nd\theta}{\cos\varphi\cos\theta}\ldots\ (4).\qquad n\frac{d\theta}{d\varphi}=\sin(\varphi-\theta)\ldots.(5).$$

Let $\omega=\phi-\theta$, then $\dfrac{d\omega}{1-(\sin\omega/n)}=d\varphi\ldots.(6)$.

Whence $\cot\left[\dfrac{\pi}{4}-\dfrac{\varphi-\theta}{2}\right]=n(\varphi+c)\ \ldots(7)$.

$\cot[\frac{1}{4}\pi-\frac{1}{2}(\tan^{-1}y-\tan^{-1}x)]=n\tan^{-1}y+c_1$, or, let

$$y=\frac{x-z}{1+xz};\text{ then }\frac{z}{[1+z^2][z+n\sqrt{(1+z^2)}]}\frac{dz}{dx}-\frac{1}{1+x^2}=0,$$

in which the variables are separated.

II. **Solution by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, O.**

Let $x=\tan\theta$, $y=\tan\varphi$, then $(\tan\varphi-\tan\theta)\sec\theta\dfrac{\sec^2\varphi}{\sec^2\theta}=n\sec^3\varphi$.

$\therefore\ \sec(\varphi-\theta)=n\dfrac{d\theta}{d\varphi}$.

Let $\omega=\varphi-\theta$, then $\dfrac{d\theta}{d\varphi}=1-\dfrac{d\omega}{d\varphi}$. $\therefore\ \sin\omega=n-n\dfrac{d\omega}{d\varphi}$.

$\therefore\ \dfrac{nd\omega}{n-\sin\omega}-d\varphi=0$.

By the formulas given in the text books, if $n>1$, numerically,

$$\frac{2n}{\sqrt{(n^2-1)}}\tan^{-1}\frac{n\tan\frac{1}{2}\omega-1}{\sqrt{(n^2-1)}}=\varphi+c_1.$$

$$\therefore\frac{2n}{\sqrt{(n^2-1)}}\,\frac{n\tan\left(\frac{1}{2}\tan^{-1}\dfrac{y-x}{1+xy}\right)-1}{\sqrt{(n^2-1)}}=\tan^{-1}y+c_1\ ;\text{ if } n<1,\text{ numerically.}$$

$$\frac{1}{\sqrt{(1-n^2)}}\log\frac{n\tan\frac{1}{2}\omega-1-\sqrt{(1-n^2)}}{n\tan\frac{1}{2}\omega-1+\sqrt{(1-n^2)}}=\varphi+c_2.$$

$$\therefore \frac{1}{\sqrt{(1-n^2)}}\log\frac{n\tan\left(\frac{1}{2}\tan^{-1}\frac{y-x}{1+xy}\right)-1-\sqrt{(1-n^2)}}{n\tan\left(\frac{1}{2}\tan^{-1}\frac{y-x}{1+xy}\right)-1+\sqrt{(1-n^2)}}=\tan^{-1}y+c_2.$$

If $n=1$, $\dfrac{d\omega}{1-\sin\omega}-d\varphi=0$.

$$\therefore \frac{d\omega}{\sin^2\frac{1}{2}\omega+\cos^2\frac{1}{2}\omega-2\sin\frac{1}{2}\omega\cos\frac{1}{2}\omega}-d\varphi=0.$$

$$\therefore \frac{d\omega\sec^2\frac{1}{2}\omega}{(\tan\frac{1}{2}\omega-1)^2}-d\varphi=0.\quad \therefore \frac{2d(\tan\frac{1}{2}\omega-1)}{(\tan\frac{1}{2}\omega-1)^2}-d\varphi=0.$$

$$\therefore \frac{2}{1-\tan\frac{1}{2}\omega}=\varphi+c_2.\quad \therefore 1+\frac{\tan\frac{1}{2}\omega+1}{1-\tan\frac{1}{2}\omega}=\varphi+c_2.$$

$$\therefore \tan(\tfrac{1}{2}\omega+\tfrac{1}{4}\pi)=\varphi+c'^2.$$

$$\therefore \tan\left[\frac{\pi}{4}+\tfrac{1}{2}\tan^{-1}\frac{y-x}{1+xy}\right]=\tan^{-1}y+c'_2\text{ ; if } n=-1,$$

$\dfrac{d\omega}{1+\sin\omega}-d\varphi=0$, and the result is $\tan\left[\dfrac{\pi}{4}-\tfrac{1}{2}\tan^{-1}\dfrac{y-x}{1+xy}\right]=\tan^{-1}y+c.$

Also solved by *L. C. WALKER*, *H. C. WHITAKER*, and *G. B. M. ZERR*.

123. Prize Problem. Proposed by B. F. FINKEL. A. M., M. Sc., Professor of Mathematics and Physics in Drury College. Springfield, Mo.

Find in finite terms, the value of $\int_0^{\frac{1}{4}\pi}\log\tan\phi d\phi$.

Solution by W. H. ECHOLS, Professor of Mathematics, University of Virginia, Charlottsville, Va.

Put $\tan z=x$. Then $\int_0^{\frac{1}{4}\pi}\log(\tan z)dz=\int_0^1\frac{\log x}{1+x^2}dx$,

$$=\int_0^1\log x(1-x^2+x^4-x^6+\dots.)dx.$$

Differentiating, respecting a, $\int_0^1 x^a dx=\dfrac{1}{a+1}$.

$$\therefore \int_0^1 x^a\log x dx=-\frac{1}{(a+1)^2}.$$

$$\therefore \int_0^{\frac{1}{4}\pi} \log(\tan z)dz = -(1-\frac{1}{3^2}+\frac{1}{5^2}-\ldots\ldots) = -\frac{\pi^3}{32},$$

by a well known summation.

A correct result was also received from Lon. C. Walker. The integration in finite terms required that the entire work should be done in finite terms as is done, for example, in Byerly's *Integral Colculus*, page 98, when

$$\int_0^{\frac{1}{2}\pi} \log\sin x dx$$

is found, and not the final result stated in finite terms.

The summation of the above series may be found by using the relation,

$$\frac{B_{2n}}{(2n)!} = \frac{2^{2n+2}}{\pi^{2n+1}}[1-\frac{1}{3^{2n}}+\frac{1}{5^{2n}}-\frac{1}{7^{2n}}+\ \ldots]$$

When $n=1$, $B_2=1$, one of Euler's numbers. See B. O. Peirce's *Table of Integrals*, page 90. ED.

124. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Show that the cardioides $r=a(1+\cos\theta)\ldots.(1)$, and $r=b(1-\cos\theta)\ \ldots(2)$, intersect at right angles.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and J. SCHEFFER, A. M., Hagerstown, Md.

The equation can be written $r=2a\cos^2\frac{1}{2}\theta$, $r=2b\sin^2\frac{1}{2}\theta$, or $r^{\frac{1}{2}}=(2a)^{\frac{1}{2}}\cos\frac{1}{2}\theta$, $r^{\frac{1}{2}}=(2b)^{\frac{1}{2}}\cos\frac{1}{2}(\pi-\theta)$.

$$\frac{dr}{rd\theta}=-\tan\tfrac{1}{2}\theta,\ \ \frac{dr}{rd\theta}=+\tan\tfrac{1}{2}(\pi-\theta).$$

At the point of intersection r and θ are the same for both.

The angle made by the perpendicular from the origin on the tangent is in the first case $\frac{1}{2}\theta$, in the second $\frac{1}{2}(\pi-\theta)$.

But $\frac{1}{2}\theta+\frac{1}{2}(\pi-\theta)=\frac{1}{2}\pi$.

$\therefore$ These perpendiculars are perpendicular to one another.

$\therefore$ The tangents are perpendicular and the cardioids intersect at right angles.

Solved substantially as above by *JOHN F. TRAVIS*, Fellow and Assistant in Mathematics in Ohio State University, Columbus, O.; *E. L. SHERWOOD, A. M.*, Professor of Mathematics, Beaver College, Beaver, Pa., and *L. C. WALKER*.

MECHANICS.

125. Proposed by T. U. TAYLOR, C. E., Professor of Civil Engineering, University of Texas, Austin, Tex.

(1) If a parabola is described on the vertical face of a reservoir wall, axis vertical and in the surface, and $P\ (h, b)$ be any point on the curve, and B the foot of the perpendicular from P on the axis, find c. p. on area OBP.

(2) If A is point where horizontal through P cuts vertical axis (OY), find c. p. on area OAP.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

From statement of problem "axis vertical" should evidently read "axis horizontal."

Let $y^2=2px$ be the parabola. But $b^2=2ph$.

$\therefore$ $y^2=b^2x/h$. Let p=pressure, m=moment, w=unit weight of water.

(1) $dp=wy(h-x)dy$, $dm=wy^2(h-x)dy$.

$$\therefore \bar{y}=\frac{\int_{-b}^{0} y^2(h-x)dy}{\int_{-b}^{0} y(h-x)dy}=\frac{\int_{-b}^{0} y^2(h-hy^2/b^2)dy}{\int_{-b}^{0} y(h-hy^2/b^2)dy}=-\tfrac{8}{15}b.$$

$$\bar{x}=\frac{\int_{0}^{h} y(h-x)^2dx}{\int_{0}^{h} y(h-x)dx}=\frac{\int_{0}^{h} \sqrt{x}\,(h-x)^2dx}{\int_{0}^{h} \sqrt{x}\,(h-x)dx}=\tfrac{4}{7}h.$$

(2) The depth of an elementary strip length x is y.

$\therefore$ $dp=wxydy$, $dm=wxy^2dy$.

$$\therefore \bar{y}=\frac{\int_{-b}^{0} xy^2dy}{\int_{-b}^{0} xydy}=\frac{\int_{-b}^{0} y^4dy}{\int_{-b}^{0} y^3dy}=-\tfrac{4}{5}b.$$

$$\bar{x}=\frac{\int_{0}^{h} x^2(b-y)dx}{\int_{0}^{h} x(b-y)dx}=\frac{\int_{0}^{h} x^2\left(b-\frac{b\sqrt{x}}{\sqrt{h}}\right)dx}{\int_{0}^{h} x\left(b-\frac{b\sqrt{x}}{\sqrt{h}}\right)dx}=\frac{\int_{0}^{h} x^2(\sqrt{h}-\sqrt{x})dx}{\int_{0}^{h} x(\sqrt{h}-\sqrt{x})dx}=\tfrac{5}{21}h.$$

126. Proposed by W. J. GREENSTREET. M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

AB is the horizontal base of a smooth cycloidal tube, vertex downward. A sphere is placed in the tube at A, and when it reaches the vertex another sphere of different mass is placed in the tube at B. When and where do they meet, and find their velocity immediately after collision, the spheres being partially elastic ?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let a be the radius of the generating circle, e the coefficient of restitution. m=mass of first sphere, m_1=mass of second sphere. It has been demonstrated that the first sphere will reach the vertex in the time, $t=\pi\sqrt{(a/g)}$, with a velocity, $v=2\sqrt{(ag)}$.

To descend half the vertical height it requires $\frac{1}{2}\pi\sqrt{(a/g)}$ seconds.

$\therefore$ The spheres meet at a vertical distance equal to the radius of the generating circle above the vertex, $\frac{1}{2}\pi\sqrt{(a/g)}$ seconds after the second sphere is placed in the tube.

At the moment of impact both spheres have a velocity$=\sqrt{(2ag)}$. Let v_1, v' be the velocity of the first and second spheres after impact. Then for direct impact $v_1-v'=e[\sqrt{(2ag)}-\sqrt{(2ag)}]=0$.

$\therefore v_1=v'$, also $\sqrt{(2ag)}(m+m_1)=mv_1+m_1v'$.

$\therefore v_1=v'=\sqrt{(2ag)}$.

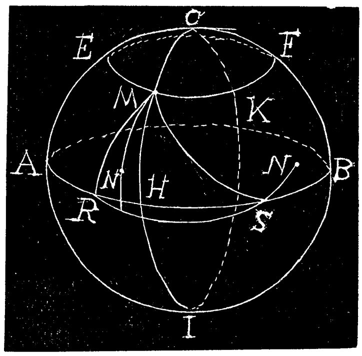
AVERAGE AND PROBABILITY.

107. Proposed by **LON C. WALKER. A. M.**, Professor of Mathematics, Petaluma High School. Petaluma. Cal.

Two points are taken at random in the curved surface of a hemisphere. Show (1) that the average length of the straight therein is $32r/9\pi$; and (2) that the average length of the arc of a great circle, which joins them, is $4r/\pi$.

Solution by **G. B. M. ZERR, A. M., Ph. D.**, Professor of Chemistry and Physics in The Temple College. Philadelphia, Pa.

Let m, n be two random points on the same side of the great circle $AHBK$. Through M draw the great circle $CHIK$ perpendicular to $AHBK$, and the small circle EMF parallel to $AHBK$. From M as a pole with a spherical radius equal to MN describe the arc RSN intersecting $AHBK$ in R and S.



Let arc $MN=r\theta$, $HM=r\varphi$, $RH=HS=r\mu$, $\angle RMH=\psi$.

Then $\sin\mu=\sin\theta\sin\psi\ldots.(1)$.

$\cos\psi=\tan\varphi\cot\theta\ldots.(2)$.

$\cos\theta=\cos\varphi\cos\mu\ldots.(3)$.

Length of straight$=2r\sin\frac{1}{2}\theta$.

An element of surface at M is $2\pi r\cos\varphi rd\varphi$.

If θ is less than $\frac{1}{2}\pi$, for each position of M, N may be taken anywhere on the supplement of the arc RNS, if φ varies from 0 to θ. If θ is greater than $\frac{1}{2}\pi$, while φ varies from 0 to $\pi-\theta$, N can be taken on the same arc. The length of this arc is $2(\pi-\psi)r\sin\theta$. For φ from 0 to $\frac{1}{2}\pi$ and θ from 0 to φ, N can be taken anywhere on the circumference $2\pi r\sin\theta$. Then

$$(1)\ \Delta=\frac{1}{4\pi^2r^4}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}2r\sin\tfrac{1}{2}\theta.2(\pi-\psi)r\sin\theta rd\theta.2\pi r\cos\varphi rd\varphi$$

$$+\frac{1}{4\pi^2r^4}\int_{\frac{1}{2}\pi}^{\pi}\int_0^{\pi-\theta}2r\sin\tfrac{1}{2}\theta.2(\pi-\psi)r\sin\theta rd\theta.2\pi r\cos\varphi rd\varphi$$

The first result is the same as the average distance of a point in a circle from a point in its circumference. The second result is the same as the average distance between two points on the circumference of a circle.

Also solved by *F. P. MATZ*. Professor Matz in his solution did not go through the details of integration as Professor Zerr has done.

II. Solution by L. C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Let P, Q be the random points, O the center of the hemisphere, and A the vertex. Put $OP=r$, $\angle POQ=\theta$, $\angle AOP=\phi$, M_1=average length of the straight line, and M_2=average length of the arc of a circle, which joins the points.

Now while θ is constant, and $<\frac{1}{2}\pi$, and $\phi<\frac{1}{2}\pi-\theta$, for each position of P, Q may be taken anywhere in the circumference of a small circle whose pole is P, and radius $r\sin\theta$.

But when $\theta<\frac{1}{2}\pi$, and $\phi>\frac{1}{2}\pi-\theta$, or when $\theta>\frac{1}{2}\pi$, and $\phi>\theta-\frac{1}{2}\pi$, for each position of P, Q may be taken anywhere in the arc of a small circle whose pole is P, and length $2r\sin\theta[\pi-\cos^{-1}(\cot\theta\cot\phi)]$.

Hence, if θ is of given value, and $<\frac{1}{2}\pi$, the number of ways the two points can be taken is

$$\int_0^{\frac{1}{2}\pi-\theta} 2\pi r\sin\theta.2\pi r\sin\phi.rd\phi+\int_{\frac{1}{2}\pi-\theta}^{\frac{1}{2}\pi} 2r\sin\theta[\pi-\cos^{-1}(\cot\theta\cot\phi)].2\pi r\sin\phi.rd\phi$$

$$=4\pi r^3(\pi-\theta)\sin\theta.$$

If $\theta>\frac{1}{2}\pi$, the number of ways the two points can be taken is

$$\int_{\theta-\frac{1}{2}\pi}^{\frac{1}{2}\pi} 2r\sin\theta[\pi-\cos^{-1}(\cot\theta\cot\phi)].2\pi r\sin\phi.rd\phi=4\pi r^3(\pi-\theta)\sin\theta.$$

Hence, since the whole number of ways the two points can be taken is $4\pi^2r^4$, we have

$$M_1=\frac{1}{4\pi^2r^4}\int_0^{\pi} 4\pi r^3(\pi-\theta)\sin\theta.2r\sin\tfrac{1}{2}\theta.rd\theta=\frac{32r}{9\pi},\text{ and}$$

$$M_2=\frac{1}{4\pi^2r^4}\int_0^{\pi} 4\pi r^3(\pi-\theta)\sin\theta.r\theta.rd\theta=\frac{4r}{\pi}.$$

MISCELLANEOUS.

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101. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

A wire is laid along the surface of a right cone semi-vertical angle β so that it cuts the generators everywhere at a constant angle θ. Find the radius of curvature and radius of torsion.

Solution by the **PROPOSER.**

Let s=length of wire from origin to any point, ρ=radius of curvature, $1/\sigma$=radius of torsion. From problem 85, Calculus, No. 4, Vol. VI., we get $x=r\cos\varphi$, $y=r\sin\varphi$, $z=s\cos\beta\cos\theta$, $\varphi=\dfrac{\tan\theta}{\sin\beta}\log s$, $r=s\sin\beta\cos\theta$, $dx=\cos\varphi dr-r\sin\varphi d\varphi$, $dy=\sin\varphi dr+r\cos\varphi d\varphi$, $dz=\cos\beta\cos\theta ds$, $d\varphi=\dfrac{\tan\theta}{\sin\beta}\cdot\dfrac{ds}{s}$, $dr=\sin\beta\cos\theta ds$.

$\therefore$ $dx/ds=\sin\beta\cos\theta\cos\varphi-\sin\theta\sin\varphi$, $dy/ds=\sin\beta\cos\theta\sin\varphi+\sin\theta\cos\varphi$,

$dz/ds=\cos\beta\cos\theta$, $d^2x/ds^2=-\dfrac{\sin\theta}{s}\left(\sin\varphi+\dfrac{\tan\theta\cos\varphi}{\sin\beta}\right)$,

$d^2y/ds^2=\dfrac{\sin\theta}{s}\left(\cos\varphi-\dfrac{\tan\theta\sin\varphi}{\sin\beta}\right)$, $d^2z/ds^2=d^3z/ds^3=0$,

$d^3x/ds^3=-\dfrac{\sin\theta\tan\theta}{s^2\sin\beta}\left(\cos\varphi-\dfrac{\tan\theta\sin\varphi}{\sin\beta}\right)$, $d^3y/ds^3=-\dfrac{\sin\theta\tan\theta}{s^2\sin\beta}\left(\sin\varphi+\dfrac{\tan\theta\cos\varphi}{\sin\beta}\right)$

$$1/\rho^2=(d^2x/ds^2)^2+(d^2y/ds^2)^2+(d^2z/ds^2)^2=\frac{\sin^2\theta}{s^2}\left(1+\frac{\tan^2\theta}{\sin^2\beta}\right).$$

$$\frac{1}{\rho^2\sigma}=\begin{vmatrix}\dfrac{dx}{ds}, & \dfrac{dy}{ds}, & \dfrac{dz}{ds}\\ \dfrac{d^2x}{ds^2}, & \dfrac{d^2y}{ds^2}, & \dfrac{d^2z}{ds^2}\\ \dfrac{d^3x}{ds^3}, & \dfrac{d^3y}{ds^3}, & \dfrac{d^3z}{ds^3}\end{vmatrix}=\frac{dz}{ds}\left(\frac{d^2x}{ds^2}\cdot\frac{d^3y}{ds^3}-\frac{d^3x}{ds^3}\cdot\frac{d^2y}{ds^2}\right)$$

$$=\frac{\sin^2\theta\tan\theta}{s^3\sin\beta}\left(1+\frac{\tan^2\theta}{\sin^2\beta}\right)\cos\beta\cos\theta.$$

$$\therefore\ 1/\sigma=\frac{\tan\theta\cos\beta\cos\theta}{s\sin\beta}=\frac{\sin\theta\cos\beta}{s\sin\beta}$$

Also solved by *WILLIAM HOOVER*.

102. Proposed by **CHARLES C. CROSS**, Whaleyville, Va.

Required the least multiple of 17 which when divided by 2, 3, 4, . . . 16 leaves in each case 1 as a remainder.

I. Solution by **MARVIN E. SMITHEY, A. M.**, Instructor in Mathematics, Randolph-Macon Academy, Bedford City, Va.

$x(2^4.3^2.5.7.11.13)+1$, x being any positive integer, represents all numbers satisfying the given condition. In order to determine what value of x will make this number the least multiple of 17, the following equation must be solved in least positive integers.

$$720720x+1=17y, \quad y=42395x+(5x+1)/17.$$

Let $(5x+1)/17=m$, an integer.

$\therefore x=(17m-1)/5$.

Make $m=3$, then $x=10$. The required number is $(10)720720+1=7207210$.

Solved in a similar manner of *L. M. COFFIN*, University of Maine, Orono, Me.; *J. CHAS. RATHBUN*, Student State University of Washington, Seattle, Wash.; and *G. B. M. ZERR*.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

The problem may be generally stated: $an\div2$, 3, 4, 5, ...$(a-1)$, respectively, gives a remainder of 1; where a=any prime odd number.

Let M=least common multiple of 2, 3, 4, 5, ... $(a-1)$.

Then $an=Mm+1$, and $n=\dfrac{Mm+1}{a}=Qm+\dfrac{dm+1}{a}$.

To render $\dfrac{dm+1}{a}$ integral, we put $\dfrac{dm+1}{a}=d(p-1)+c$, where p=any integer, and $c=n$ in the *first multiple* of a that renders m integral in the equation $dm+1=ac$.

When $m=a(p-1)+\dfrac{ac-1}{d}$; or the least value of m is $\dfrac{ac-1}{d}$, and every subsequent value of m is a greater than the value just preceding.

$\therefore an=M[a(p-1)+\dfrac{ac-1}{d}]+1$; the least value being obtained when $p=1$.

Also solved by *JOHN G. KELLAR*, *HARRY S. VANDIVER*, *L. C. WALKER*, *H. C. WHITAKER*, and the *PROPOSER*.

103. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics, Boys' High School, Reading, Pa.

Solve, $\log\sin x=\sin\log x$.

Solution by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

1. Suppose that imaginary values are neglected; then $\sin x$ must be positive and therefore x is $2n\pi$ and $(2n+1)\pi$, n being a whole number either positive, negative, or zero.

2 Since $\sin x$ is less than 1, $\log\sin x$ is negative, therefore $\sin\log x$ is also negative and hence $\log x$ is between $(2m+1)\pi$ and $2(m+1)\pi$; taking $m=0$, the limits of x for this condition are e^{π} and $e^{2\pi}$ or 23.146 and 535.5. By the first condition, numbers between 23 and 535.5 when divided by 6.2832 must leave a remainder less than 3.1416; the least positive value of x satisfying both conditions is then found to be between 25.1328 and 26.704. By trial this least positive value of x is found to be 26.208. Another value exists between 26.704 and 28.274 (about 27.4 by trial). Another value between 31.416 and 34.557, and so on, the successive limits for each new value of x being found by adding 6.2832 to the limits already found until 535.5 is reached. After that no real values of x are obtainable until 12392 $(=e^{3\pi})$ is reached; then two more will occur for each increase of 6.2832 until 286751 $(=e^{4\pi})$ is reached—and so on.

In a similar manner, negative values of x may be considered.

Also solved by *G. B. M. ZERR*.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

152. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

An operator on 'Change gains 5% on his *daily capital* every *odd* day of a business-week, and loses 5% of the same capital every *even* day of same week. What per cent. of his *original capital* will he have gained, or lost, at the end of a business-week ?

153. Proposed by J. C. CORBIN, Pine Bluff, Ark.

Find some two-figure numbers, such that if they be squared, then the figures interchanged and the resulting numbers squared, the resulting products will consist of the same digits in reversed order.

ALGEBRA.

153. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Eliminate x, y, z from the equations,

$$x^2+yz=a,$$
$$y^2+xz=b,$$
$$z^2+xy=c,$$
$$x+y+z=0.$$

154. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Show that the equation, $x^4+qx^2+s=0$ ...(1), can not have three *equal* roots.

155. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

If the roots of the cubic $x^3+3px^2+3qx+r=0$ be in harmonical progression, $2q^3=r(3pq-r)$.

156. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

$(z+x)a-(z-x)b=2yz.$

CALCULUS.

145. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Find the surface bounding the volume required in problem 102.

146. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

$$I=\int_0^{\frac{1}{2}\pi}\sqrt{[(p+\tfrac{1}{4}q^2)-(\tfrac{1}{2}q+\cos\theta)^2]}\,d\theta=\text{what ?}$$

147. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Find volume common to the two solids

$$(x/a)^{\frac{2}{3}}+(y/b)^{\frac{2}{3}}=(z/c)^{\frac{3}{8}}\,, \quad (y/b)^{\frac{2}{3}}+(z/c)^{\frac{2}{3}}=(x/a)^{\frac{1}{3}}$$

MECHANICS.

136. Proposed by F. T. WRIGHT, Ph. B., Schenectady, N. Y.

In an air brake test a train moving at 22 miles an hour on a down grade of one per cent. was stopped in 91 feet. There was 94 per cent. of the train braked. Taking the fractional resistance as 8 pounds per ton, find the net brake resistance per ton.

137. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

A uniform inextensible string rests against the inner side of a smooth elliptic wire semi-axes a and b, and is repelled from the foci and the center by the following forces: μ/rd and $\nu/r'd$ emanating from the foci, and $\pi c/d$ from the center, the distances of any point on the string from the foci being r and r', respectively, its distance from the center being c, and the semi-conjugate diameter corresponding to the point being d. Find the pressure on the wire at any point.

DIOPHANTINE ANALYSIS.

99. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

Find a general expression for the radius of the sphere which, dropped in (or partly in) a right cone full of water, will displace the most water; the radius of the sphere, and the width, height and slant height of the cone to be rational integral numbers.

100. Proposed by LON C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

(a) Find the least three integral numbers such, that if to the square of each the product of the other two be added, the three sums shall be all squares.

(b) Find the two least integral numbers such, that not only each of them, but also their sum and their difference, when increased by *unity*, shall be all square numbers.

AVERAGE AND PROBABILITY.

121. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Find the average area of the pentagon formed by joining five random points on the surface of a given circle.

122. Proposed by F. M. PRIEST, St. Louis, Mo.

Suppose each of the nine digits to be placed in a wheel, and five of them drawn at random therefrom, and written down in the order drawn. What is the probability the number thus expressed will be greater than 50,000?

MISCELLANEOUS.

121. Proposed by F. P. MATZ, Sc. D.. Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

How can we determine the elements of a cyclone from observations made at three different points?

122. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

How should a Division of Space be made in order that the *Partional Area* may be a Minimum?

123. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

If a *curve* of the third degree can not be made to pass through more than six arbitrarily chosen points, why can a *surface* of the third degree be made to pass through nineteen such points?

NOTES.

THE MONTHLY is mailed on the 28th of each month and should reach its subscribers soon after that time.

The notice of the new Postal ruling which appeared in our last issue was somewhat premature. We had received a number of notices to that effect, but none of them were official.

This number of THE MONTHLY is sent to all of our old subscribers who have not yet renewed their subscriptions as well as to those who have renewed. Any one who does not wish to continue his subscription will please return this number with his name plainly written on the wrapper.

Owing to the demands on his time in the preparation of other mathematical text-books, of which the Colaw and Ellwood Arithmetics form a part, Professor Colaw's connection with THE MONTHLY will be temporarily severed. In view of this fact all contributions should be sent to B. F. Finkel until further notice is given.

We desire to thank all of our subscribers and contributors for the many kind words that have come to us to encourage us in our work. We also desire to thank all of our subscribers who so promptly responded to our notices for renewals. To renew promptly is one of the easy ways in which every one can very materially assist us in carrying on the good work of THE MONTHLY, for it should be remembered that our financial problem is one which gives us much anxiety.

Dr. William Anthony Granville, Instructor in Mathematics in Sheffield Scientific School of Yale University, has designed a plotting paper, The Polar Coördinate Plotting Paper. Every teacher of Analytical Geometry must often have felt the need of such paper. But not only is this paper serviceable to the students of Analytical Geometry and Calculus, but also in the solution of problems in Vector Analysis, Mechanics, Astronomy, and Engineering. The price of the paper is about the same as for ordinary rectangular plotting paper. Persons desiring to use this paper should write to Dr. Granville.

BOOKS.

An Elementary Treatise on the Calculus, with illustrations from Geometry, Mechanics, and Physics. By George A. Gibson, M. A., F. R. S. E., Professor of Mathematics in the Glasgow and West of Scotland Technical College. Cloth, 12mo., xix+459 pages. Price, $1.90, net. New York: The Macmillan Co.

In writing this work, the aim of the author seems to have been to prepare the student for immediately applying the principles and processes of the Calculus in any department of his studies in which the Calculus is used. With this end in view he has illustrated the applications of the Calculus by drawing on Geometry, Mechanics, and Physics. We heartily approve of this method of treating the subject. The Calculus is being studied by a larger number of students to-day than ever before, and by its making use of illustrations in such apparently unrelated subjects as Physics and Chemistry, greater interest is aroused. This book emphasizes the fact that even in such a subject as Chemistry a sound knowledge of the Calculus is of especial importance, since it is the properties of functions of more than one variable that are predominant in chemical investigations. The book closes with a short chapter on Ordinary Differential Equations, designed to illustrate the types of equations most frequently met with in dynamics, physics, and mechanical and electrical engineering.

B. F. F.

The Groups of Steiner in Problems of Contact, by Dr. Leonard E. Dickson. Reprinted from the Transactions of the American Mathematical Society.

Representation of Linear Groups as Transitive Substitution Groups. By Dr. Leonard E. Dickson. Reprinted from the American Journal of Mathematics, Vol. XXIII, No. 4.

THE AMERICAN MATHEMATICAL MONTHLY.

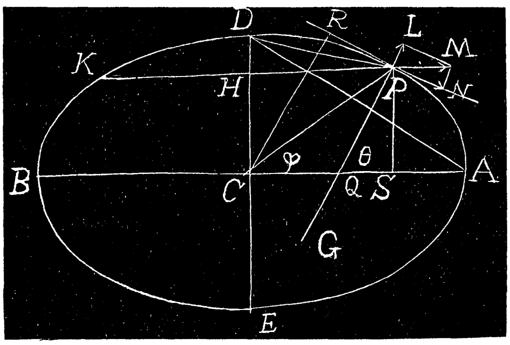
Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. IX. FEBRUARY, 1902. No. 2.

GRAVITY, TRUE AND APPARENT.

By G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.

Let P be any point on the earth's surface, $CA=CB=a$, the earth's equatorial radius, $CD=CE=b$, the earth's polar radius, (x, y) the coördinates of P, $CP=r$, $\angle PQA=\theta=$latitude of P, $\angle PCA=\varphi$, mass of particle at $P=$unity, $f=$ centrifugal force of particle at P, $e=$earth's eccentricity, $g'=$true gravity in any latitude, $g=$apparant gravity in any latitude, $G'=$true gravity at the equator, $G=$apparant gravity at the equator, $l'=$length of second's pendulum for true gravity in any latitude, $l=$length of second's pendulum for apparent gravity in any latitude, $L'=$length of second's pendulum for true gravity at the equator, $L=$length of second's pendulum for apparent gravity at the equator, $T=$number of seconds in a siderial day.



Then $HP=r\cos\varphi=x$, $CH=r\sin\varphi=y$, $HP=CQ+QS$, $CQ=e^2x=e^2r\cos\varphi$.

$$PQ=\frac{b}{a}\sqrt{(a^2-e^2x^2)}=\sqrt{(b^4/a^2+e^2y^2)}=\frac{b}{a}\sqrt{(a^2-e^2r^2\cos^2\varphi)}$$

$$=\sqrt{(b^4/a^2+e^2r^2\sin^2\varphi)},\ QS=PQ\cos\theta.$$

$$\therefore\ r\cos\varphi=e^2r\cos\varphi+\frac{b\cos\theta}{a}\sqrt{(a^2-e^2r^2\cos^2\varphi.)}$$

$$\therefore\ r\cos\varphi=\frac{a^2\cos\theta}{\sqrt{(b^2+a^2e^2\cos^2\theta)}}=\frac{a\cos\theta}{\sqrt{(1-e^2\sin^2\theta)}}.$$

$$PS=CH=r\sin\varphi=PQ\sin\theta=\sin\theta\sqrt{(b^4/a^2+e^2r^2\sin^2\varphi)}.$$

$$\therefore\ r\sin\varphi=\frac{b^2\sin\theta}{a\sqrt{(1-e^2\sin^2\theta)}}=\frac{b\sqrt{(1-e^2)}\sin\theta}{\sqrt{(1-e^2\sin^2\theta)}}.\quad f=v^2/HP=\frac{4\pi^2HP}{T^2}.$$

The component of f along PQ is $PL=f\cos\theta$.

$$\therefore\ f\cos\theta=\frac{4\pi^2HP\cos\theta}{T^2}=\frac{4a\pi^2\cos^2\theta}{T^2\sqrt{(1-e^2\sin^2\theta)}}.$$

According to the spheroid of Listing of Göttingen, 1878, a=6377377 meters=20923536 feet, b=6355270 meters=20851005 feet.

$$\therefore\ e^2=\frac{a^2-b^2}{a^2}=.00692.\quad \text{But } T=86164 \text{ seconds.}$$

$$\therefore\ f\cos\theta=\frac{.1112553\cos^2\theta}{\sqrt{(1-e^2\sin^2\theta)}}=.1112553[1-(1-\tfrac{1}{2}e^2)\sin^2\theta] \text{ nearly,}$$

$$=.1112553(1-.99654\sin^2\theta) \text{ feet per second.}$$

According to Newcomb and Holden the true gravity is given by $g'=G'(1+.00173\sin^2\theta)$ for any latitude.

Now $g'-f\cos\theta=g$, also $G'=32.2015235$.

$\therefore\ g=32.2015235(1+.00173\sin^2\theta)-.1112553(1-.99654\sin^2\theta)$

$=32.0902682(1+.00519\sin^2\theta)=G(1+.00519\sin^2\theta)$.

$.00173=\frac{1}{4}e^2,\ .00519=\frac{3}{4}e^2$.

$\therefore\ g'=G'(1+\frac{1}{4}e^2\sin^2\theta),\ g=G(1+\frac{3}{4}e^2\sin^2\theta)$.

In all previous writings these formulae are given in the form $g=g_0(1+c\sin^2\theta)$ where c is a constant. Whether this constant is a function of any dimension of the spheroid or whether it has any connection with the spheroid is not stated. I here express both constants in terms of the eccentricity. Dr. Mendenhall, the greatest American authority on gravity, has told me that these constants were never before so expressed.

$$l' = g'/\pi^2 = \frac{32.2015235}{9.8696044}(1+\tfrac{1}{4}e^2\sin^2\theta) = 3.2626965(1+\tfrac{1}{4}e^2\sin^2\theta) \text{ feet,}$$

$$=39.1523578(1+\tfrac{1}{4}e^2\sin^2\theta) \text{ inches} = 994.45(1+\tfrac{1}{4}e^2\sin^2\theta)mm.$$

$$\therefore\ l' = L'(1+\tfrac{1}{4}e^2\sin^2\theta).$$

$$l = g/\pi^2 = \frac{32.0902682}{9.8696044}(1+\tfrac{3}{4}e^2\sin^2\theta) = 3.251424(1+\tfrac{3}{4}e^2\sin^2\theta) \text{ feet}$$

$$=39.017088(1+\tfrac{3}{4}e^2\sin^2\theta) \text{ inches} = 991.01(1+\tfrac{3}{4}e^2\sin^2\theta)mm.$$

$$\therefore\ l = L(1+\tfrac{3}{4}e^2\sin^2\theta).$$

To find the time of revolution of the earth on its axis so that centrifugal force may equal gravity.

First, at the equator.

Let A be the bob of a conical pendulum, $DC=b=$height of cone generated by this pendulum. The formula for time of revolution of this pendulum is

$$t = 2\pi\sqrt{\frac{b}{G'}} = 2\pi\sqrt{\frac{12b}{\pi^2 L'}} = 2\sqrt{\frac{12b}{L'}}$$

where L' is given in inches, and b in feet.

$$\therefore\ t = 2\sqrt{\frac{20851005}{3.2626965}} = 5056 \text{ seconds.}$$

At present the earth revolves once 86164 seconds. $86164 \div 5056 = 17.04$ times as fast as at present.

Second, in latitude 30°.

Let P be the bob of the pendulum.

Then $t = 2\pi\sqrt{\dfrac{DH}{g'}} = 2\sqrt{\dfrac{12DH}{l'}}$, l' in inches, DH in feet.

$$DH = b - CH = b - r\sin\varphi = b - \frac{b\sqrt{(1-e^2)}\sin\theta}{\sqrt{(1-e^2\sin^2\theta)}} = b\left(1-\sqrt{\frac{1-e^2}{4-e^2}}\right), \text{ since } \theta=30°.$$

$$\therefore\ DH = .5013b = 10452608.8065 \text{ feet.}$$

$$l' = 39.1523578(1+\tfrac{1}{16}e^2) = 39.16929336 \text{ inches.}$$

$$\therefore\ t = 2\sqrt{\frac{10452608.8065}{3.26410778}} = 3579 \text{ seconds.}$$

86164÷3579=24.075 times as fast as at present.

∴ One revolution per hour.

By this method we can find the time of revolution for any latitude. It is

$$t=4\sqrt{\frac{3b}{L'}}\left\{\frac{\left(1-\frac{\sqrt{(1-e^2)}\sin\theta}{\sqrt{(1-e^2\sin^2\theta)}}\right)}{(1+\frac{1}{4}e^2\sin^2\theta)}\right\}^{\frac{1}{2}}$$

where b is taken in feet, and L' in inches.

The following table gives gravity and the length of the second's pendulum for every five degrees under existing conditions.

DEGREES	GRAVITY		SECOND'S PENDULUM	
	POUNDALS	DYNES	INCHES	*mm*
0°	32.0002682	978.09	39.017088	991.01
5°	32.0915197	978.13	39.018610	691.05
10°	32.0952903	978.25	39.023194	991.17
15°	32.1014356	978.43	39.030666	991.36
20°	32.1097470	978.69	39.040771	991.62
25°	32.1200159	979.00	39.053257	991.93
30°	32.1319053	979.36	39.067713	992.30
35°	32.1450623	979.76	39.083710	992.71
40°	32.1590697	980.19	39.100741	993.14
45°	32.1735424	980.63	39.118337	993.59
50°	32.1880023	981.07	39.135918	994.03
55°	32.2020226	981.50	39.152965	994.47
60°	32.2151796	981.90	39.168962	994.87
65°	32.2270722	982.26	39.183422	995.24
70°	32.2373379	982.57	39.195903	995.56
75°	32.2456621	982.83	39.206024	995.81
80°	32.2517955	983.02	39.213482	996.00
85°	32.2555651	983.13	39.218065	996.12
90°	32.2568167	983.17	39.219587	996.16

A PROBLEM CONNECTED WITH MERSENNE'S NUMBERS.

By HARRY S. VANDIVER, Bala, Pa.

In Sir W. R. Ball's *Recreations and Problems* (London, 1809), page 33, I find the following:

"A curious proposition which comes from China, and which I believe appears here in print for the first time, is *that* $(2^n-2)/n$ *is an integer if* n *is a prime and is not an integer if* n *is not a prime.* The first of these statements is at once demonstrable; but I have not succeeded in proving the second part of the propo-

sition, which seems to introduce considerations similar to those involved in the theory of Mersenne's Numbers."

This proposition is of extraordinary importance, since, if we suppose it true then it gives a complete analytic definition of a prime. For instance, to find whether or not a is a prime it would only be necessary to calculate the residue of 2^{a-1} with respect to a. If we find $2^{a-1}\equiv 1(\text{mod} a)$ then a is a prime, but if 2^{a-1} is not congruent to 1 (mod$a$) then a is composite.

The object of the following investigation is to show that the second part of the proposition is *false*.

To prove this falsity it is sufficient to find a value n=an odd composite such that $2^{n-1}\equiv 1(\text{mod} n)$......(1).

Let us suppose, first that n is the simplest form of odd composite=$p\times q$, where p and q are primes. Then, since $\phi(n)=(p-1)(q-1)$, we have

$$2^{(p-1)(q-1)}\equiv 1(\text{mod} pq)$$

by Fermat's Generalized Theorem. If we assume that m is the smallest number such that $2^m\equiv 1(\text{mod} pq)$, then m must be a divisor of $\phi(n)$ and if (1) is possible then n must be a multiple of m. (Serret's *Algebra*, Sup. Vol. 2, page 48).

These conditions may be written

$$pq-1\equiv 0(\text{mod} m)$$
$$(p-1)(q-1)\equiv 0(\text{mod} m)$$

Comparing these congruences, we find

$$p\equiv 1(\text{mod} m), \quad q\equiv 1(\text{mod} m).$$

Then to prove the possibility of (1) it is sufficient to find values, m, p, q, such that $2^m\equiv 1(\text{mod} pq)$ where $q\equiv 1(\text{mod} m)\equiv q$ and m is not greater than $\phi(pq)$.

To find these values, assign to m small integral values in succession to find, by trial, numbers for p and q corresponding.

If m=1 to 9 inclusive, no appropriate values can be found for p and q, but if m=10, then we can put

$$p=10\times 1+1 \text{ and } q=3\times 10+1,$$

and then $$2^{10}\equiv 1(\text{mod} 31\times 11)$$

and therefore $$2^{340}\equiv 1(\text{mod} 31\times 11)$$

and the second part of the proposition originally quoted is thus seen to be false.

If m=11, then putting p=23, q=89,

$$2^{2046}\equiv 1(\text{mod} 2047).$$

In the same manner we can find other values of m, p, and q, and the number of possible sets does not appear to be limited.

The first part of the proposition, namely, $(2^n-2)/n$ is an integer when n is prime, is but a particular case of Fermat's Theorem that $a^{p-1}\equiv 1(\text{mod}\,p)$ when p is prime and a is prime to p.

Bala, Pa., Feb. 1, 1902.

GEOMETRIC DERIVATION OF CERTAIN TRIGONOMETRIC FORMULAE.

By PROFESSOR L. E. DICKSON.

Students of trigonometry find it interesting to have, in addition to the usual proof, the following geometric derivation of the formulae used in the solution of a plane triangle of given sides. The only trigonometry used is the *definition* of the tangent ratio.

The first step is the usual geometric proof (by means of the theorem giving, in terms of the sides, the projection of one side on another side) of Heron's formula:

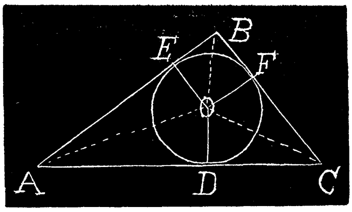
$$\triangle=\sqrt{[s(s-a)(s-b)(s-c)]}.$$

The next step is the evaluation of the radius r of the circle inscribed in the triangle. It its center is O, the areas of the triangles AOB, BOC, COA are $\frac{1}{2}cr$, $\frac{1}{2}ar$, $\frac{1}{2}br$, respectively. Hence

$$\triangle=\tfrac{1}{2}(a+b+c)r=sr.$$

$$\therefore\ r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

The next step is the simple proof that the length of the tangent from A to the inscribed circle is $AE=AD=s-a$; from B, $BE=BF=s-b$, and from C, $CF=CD=s-c$. Then



$$\tan\tfrac{1}{2}A=\frac{r}{s-a},\ \tan\tfrac{1}{2}B=\frac{r}{s-b},\ \tan\tfrac{1}{2}C=\frac{r}{s-c}.$$

These are the most convenient formulae for the solution of a triangle of given sides. We may, however, derive at once the formulae for $\tan\frac{1}{2}A$, $\sin\frac{1}{2}A$, $\cos\frac{1}{2}A$ in terms of a, b, c only. Evidently,

$$\tan\tfrac{1}{2}A=\sqrt{\frac{(s-b)\ (s-c)}{s(s-a)}}.$$

We may obtain $\cos\frac{1}{2}A$ by employing the formula

$$\sec^2\tfrac{1}{2}A=1+\tan^2\tfrac{1}{2}A=1+\frac{(s-b)\ (s-c)}{s(s-a)}=\frac{2s^2-s(a+b+c)+bc}{s(s-a)}=\frac{bc}{s(s-a)};$$

or from the figure, since $AO^2=r^2+(s-a)^2$ reduces to $bc(s-a)/s$. Then

$$\cos\tfrac{1}{2}A=\sqrt{\frac{s(s-a)}{bc}},\ \sin\tfrac{1}{2}A=\tan\tfrac{1}{2}A\cos\tfrac{1}{2}A=\sqrt{\frac{(s-b)\ (s-c)}{bc}}.$$

Similarly, a proof of the sine proportion follows from the relation $a/\sin A$ $=D=$constant$=$diameter of circumscribed circle.

University of Chicago, Jan. 1902.

SOME FALLACIES IN WENTWORTH'S GEOMETRY.

By DR. GEORGE BRUCE HALSTED.

Often both to teacher and pupil instruction seems satisfactory which to the expert is deliciously absurd.

The American-University-girl's French in Paris is a field of rich humor. Fashionable restaurants have hired waiters for their ability to understand this strange language, American French.

I went once with a party of Americans to buy gloves at a famous place on the Avenue de la Opera. "How does it happen that you understand our French so well?" was asked the attendant.

"Why, that's not strange," she answered, "I am an English-woman!"

Again, a young lady from Texas, a student of French, bought a parasol at the Bon Marché, that store which has the genius to say "We will exchange what you buy, whenever it ceases to please you." The very next day she took it back, but what she really did say in French to the attendant was "I bought this here *tomorrow*." The polite attendant never smiled, but, though he spoke English fluently, answered slowly in French: "If mademoiselle bought it here tomorrow, we are glad to receive it back or exchange it."

The French words for horses (chevaux) and for hair (cheveux) differ only in the one vowel, and Americans are vowel deaf.

This explains the apparently mysterious statement of the young *coiffeur* who said, "I begin to learn that American French. When an American lady drives up and asks me to wash her horses, that means she wishes her hair dressed."

A gushing southern young lady who felt sure of the word 'fiancé' for 'engaged', astounded a young hack-man by asking him in French, "Are you betrothed?" and on his answering "no," said: "Then I take you." He was so surprised that when asked his fare, always a mistake in Paris, he told the truth, but when it came to pay, asked thrice as much; when the girl exclaimed in American French, "You are more dear to me now than when we were first engaged."

Our books in elementary mathematics are American French, deliciously absurd to the expert.

Almost the only geometry used is that of my friend the New Hampshire banker, Wentworth, who begins with a mental confusion between a terminated piece of a straight line, which completely satisfies his newest definition for a straight line, and the straight line itself. This confusion is emphasized by his "Postulate: A straight line can be produced."

If this is what he means, it is a definite magnitude, like an inch or a yard. But then about it his axiom is not true.

Axiom: "Two points *determine* a straight line."

On the same page 8, is "§49. Axiom. A straight line is the shortest line that can be drawn from one point to another."

But as the great Hilbert well says: "Such an axiom is senseless, if one has not defined the concept of length of a curve."

So it need not surprise us that in attempting to attain to the length of the simplest curve, the circle, Mr. Wentworth falls into a shocking *non-sequitur*, which blunder is now on page 95 of his revised edition, 1899, and is being duly taught to thousands upon thousands of our young Americans, and by them duly swallowed as mathematics.

On page 9 he defines an angle as an *opening* [the italics are his].

On page 12, he is evidently standing on his head, since he talks upside down when he says "the magnitude of the angle depends upon the amount of rotation of the line."

But anyone who even contemplates owning a watch should have realized that it is the angle which determines the amount of rotation.

The demonstrations of even his simplest theorems are fallacious or beg the question.

Thus to prove "In an isosceles triangle the angles opposite the equal sides are equal," he uses a hypothetical construction on page 36 which he does not justify by any reference for the obvious reason that such reference would be to page 115 of his own book.

In what he gave as proof of the theorem: "Two triangles are equal when the three sides of the one are equal respectively to the three sides of the other," up to 1886 he said, page 44, "place triangle $A'B'C'$ in the position $AB'C$, having its *greatest* side $A'C'$ in coincidence with its equal AC, and its vertex at B', opposite B. Draw BB' intersecting AC at H."

Of course he did not prove this intersection, since it is enormously harder to prove than the whole theorem. But its standing unjustified must have been a

source of trouble, since in his later editions it is simply omitted, though as before his proof depends upon it.

His treatment of tangent circles has always been consistently fallacious. Two circles are called tangent when they have only one point in common. It follows then as a theorem that the line of centers passes through the point of contact. Wentworth has this theorem, page 91, but proves it by assuming it in his definition, page 75, §221. "Two circles are tangent to each other, *if both are tangent to a straight line* at the same point," which is of course only another form of the corollary: a perpendicular to the center-straight through the point of contact is a common tangent to the two circles.

The rest of the book is equally vulnerable, but, not thrice to slay the slain, we desist.

University of Texas.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

152. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

An operator on 'Change gains 5% on his *daily capital* every *odd* day of a business-week, and loses 5% of the same capital every *even* day of same week. What per cent. of his *original capital* will he have gained, or lost, at the end of a business-week?

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; HON. J. H. DRUMMOND, LL. D., Portland, Me.; and J. SCHEFFER, A. M., Hagerstown, Md.

Let 100%=original daily capital.

$100\%\times1.05=105\%$=capital at beginning of second day.

$105\%\times.95=99.75\%$=capital at beginning of third day, and so on during the week.

$\therefore$ Saturday night he would have $100\%\times(1.05)^3\times(95)^3=(99.75)^3=99.25187\%$.

$\therefore$ $100\%-99.25187\%=\frac{3}{4}\%$ nearly, his loss.

ALGEBRA.

130. Proposed by J. MARCUS BOORMAN, Woodmere, N. Y.

Solve $x^5-y^5=2101$......(1), $x-y=1$......(2). Find general formula for (1), ...(2), when $x^n-y^n=a$, $x-y=b$; for $n_0=3$; $n_1=5$; $n_2=7$, etc.

I. Solution by G.B.M.ZERR, A.M., Ph.D., Professor of Chemistry and Physics in The Temple College. Philadelphia, Pa.

$(y+1)^5-y^5=2201.$

$(y^2+y)^2+y^2+y=420.$

$\therefore\ y^2+y=20$ or $-21.$

$y=4$ or -5 or $-\frac{1}{2}(1\mp\sqrt{-83}).$

$x=5$ or -4 or $\frac{1}{2}(1\pm\sqrt{-83}).$

II. Solution by the PROPOSER.

$5=n$, $\triangle=1$. $\therefore$ Table (I), below, $1^5+5xy.1^3+5x^2y^2\,(1)=2101$; so therefore $5[x^2y^2.1^1+xy.1^3]=2101-1^5$.

$\therefore\ \frac{1}{5}[2101-1^5]=420$; hence $x^2y^2+xy+\frac{1}{4}=420+\frac{1}{4}$......(3), as $\triangle=1$, here.

$\therefore\ xy=-0.5\pm20.5$, and $4xy=80$ or -84.

$$(2)......(x-y)^2=\underline{1}\qquad\underline{1}$$

$$\therefore\ (x+y)^2=81 \text{ or } -83.$$

$\frac{1}{2}$(the root of (4)+(2)= $\qquad x=\frac{1}{2}[1\pm9]$ or $0.5[1\pm i\sqrt{83}$......(5).

$\frac{1}{2}$(the root of (4)−(2)= $\qquad y=\frac{1}{2}[-1\pm9]$ or $0.5[-1\pm i\sqrt{83}$, the eight roots.

That is, $x=5$ to $y=4$; $x=-4$ to $y=-5$; etc., as at (5).

Power Difference Theorems. Given $\triangle=x-y=b$, and any one of $w=xy$, or x^n-y^n, or $x^m+y^m,=a$, to find all by *quadratic* to $n=5$; *cubic* to $n=9$; *quintic* to $n=11$, etc. n=odd, viz., 3...5...7...9...11, etc.; m=even integers.

$$\text{(I)}.\ x^n-y^n=\Delta^n+nw\Delta^{n-2}+[\tfrac{1}{2}(n-1)(n-2)-(n-2)^0]w^2\Delta^{n-4}+$$
$$n[\tfrac{1}{6}(n-1)(n-2)-(n-3)]w^3\Delta^{n-6}+n[\tfrac{1}{24}(n-6)^3-(n-6)]w^4\Delta^{n-8}+$$
$$n[\tfrac{1}{120}(n-6)\{(n-8)^3-(n-8)\}]w^5\Delta^{n-10}+\ldots\ldots$$
$$+\tfrac{1}{24}(n^3-n)w^{\frac{1}{2}(n-3)}\Delta^3+nw^{\frac{1}{2}(n-1)}\Delta.$$

$$\text{(II)}.\ x^m+y^m=\Delta^m+mw\Delta^{m-2}+[\tfrac{1}{2}(m-1)(m-2)-(m-2)^0]w^2\Delta^{m-4}+\ldots\ldots$$
$$(\text{like } n)\ldots\ldots+\tfrac{1}{4}(m^2)w^{\frac{1}{3}(m-2)}\Delta^2+2w^{\frac{1}{2}m}.$$

Also solved by *H. C. WHITAKER.*

131. Proposed by HARRY S. VANDIVER, Bala, Pa.

It is well known that, when we define the symbol $\sqrt[n]{a}$ after the manner of elementary text-books on algebra, certain *irrational equations* may be written down which have no real or imaginary roots. Required then, the condition, if any, between a, b, c, and d such that the equation, $ax+b+\sqrt{(cx^2+d)}=0$, shall have no root, real or imaginary.

I. Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$(a^2-c)x^2+2abx=d-b^2$, or $(c-a^2)x^2-2abx=b^2-d$.

$$\therefore x=\frac{1}{c-a^2}[ab\pm\sqrt{(a^2d+b^2c-cd)}].$$

If $c>a^2$ there is no root satisfying the equation.

If $c=a^2$, $x=\dfrac{d-b^2}{2ab}$.

Then if $d>b^2$ no root will satisfy the equation.

Whenever x is positive no root can be found.

II. **Solution by the PROPOSER.**

In connection with this problem it will be well to review the subject of irrationals somewhat.

The symbol $\sqrt[n]{a}$ where a is a positive rational number will be defined as the positive rational number satisfying the relation $x^n=a$. Making use of the theory of binomial equations, then the other roots of $x^n=a$, can be represented by $\omega_n \sqrt[n]{a}$, $\omega_n^2 \sqrt[n]{a}$......$\omega_n^{n-1} \sqrt[n]{a}$, ω_n being a primitive root of $x^n=1$.

From this $\sqrt[n]{(a+bi)}$ can be defined, a and b being rational and $i^2=-1$ for $a+bi=\rho\text{cis}\theta$ ($\rho\text{cis}\theta=\rho(\cos\theta+i\sin\theta)$ of the Argand diagram, and $\sqrt[n]{(a+bi)}=\sqrt[n]{(\rho\text{cis}\theta)}\pm\text{cis}\theta/n \sqrt[n]{\rho}$, which is intelligible since

$$\sqrt[n]{\rho}=\sqrt[n]{[\sqrt{(a^2+b^2)}]}=\sqrt[2n]{(a^2+b^2)},$$

which symbol has been defined.

The above definitions treat the irrational symbol as one-valued and agree with those given in most of the recent text-books on algebra. If we define the radical symbol as multi-valued, the irrational equation loses all its individuality and might just as well be termed a rational equation.

It is a well known fact that there are radical equations which possess no roots real or imaginary, but a thorough classification of such equations has never been given, so far as I am aware. In the course of the following solution this matter will be cleared up.

Consider the simplest possible irrational equation:

$$\sqrt{x}=1......(1).$$

This possesses one root $x=1$, but the related equation, $\sqrt{x}=-1......(2)$, does not possess a root, which may be rigorously demonstrated as follows.

The equation $(\sqrt{x}+1)(\sqrt{x}-1)=0......(3)$, must contain all the roots of (1) and (2) for a value of x which reduces either factor to zero, reduces the product to zero. Expanding (3) there is obtained

$$x-1=0, \text{ or } x=1,$$

which satisfies (1) but not (2). Hence $\sqrt{x}+1=0$ possesses no root, and is the simplest irrational equation of this type.

Proceeding to the case of the equations

$$ax+b+\surd(cx^2+d)=0......(4),$$
$$ax+b-\surd(cx^2+d)=0.,....(5).$$

Put $ax+b=y$, then (4) becomes

$$-\sqrt{c\left(\frac{y-b}{a}\right)^2+d}=+y......(6),$$

and (5) becomes

$$\sqrt{c\left(\frac{y-b}{a}\right)^2+d}=+y......(7).$$

If (6) is to be satisfied by a value of y, this value cannot be positive since $\surd[f(y)]+1=0$ is an impossible relation as was proved in the case of (2). The product of (6) and (7) becomes, after simplifying,

$$(c-a^2)y^2+2bcy+a^2d+b^2c=0......(8).$$

Now it is evident that if (8) gives a positive value for y then (6) is not satisfied but (7) is; if (8) gives a negative root the reverse is the case.

In this way the following criteria are deduced, assuming that a^2d+b^2c-cd is positive. Putting

$$A=\frac{2bc}{c-a^2}, \text{ and } B=\frac{a^2d+b^2c}{c-a^2}.$$

If $A>0$ and $B>0$, then (4) has no roots and (5) has two roots.

If $A>0$ and $B<0$, or $A<0$ and $B<0$, then (4) has one root and (5) has one root.

If $A<0$ and $B>0$, then (4) has two roots and (5) has no roots..

This solves the problem as originally stated. In an analogous way one could deduce criteria for the solubility of irrational equations of any type whatever.

This problem was also discussed by J. M. Boorman, who arrived at the conclusion that there is no condition for the equation to have no roots. He maintains that Newton's (his) theorem "Every algebraic equation has as many roots as it has dimensions," holds good. But this is not true. D'Alembert's theorem does not apply in this case at all since the equation is not a rational algebraic equation. It may not be out of place to state here D'Alembert's theorem in full: "Every rational, integral, algebreic equation whose coefficients are imaginary quantities, or, in special cases, real quantities, has at least one root." When this theorem is proved it follows as a corollary that there are as many roots as there are units in the number expressing the degree of the equation. ED.

132.. Proposed by J. SCHEFFER. A. M., Hagerstown, Md.

Solve $2^x+3^y=4$; $5^x+6^y=7$.

Solution by H. C. WHITAKER. C. E., Ph. D., Manual Training School, Philadelphia, Pa.; MARCUS BAKER, U. S. Geological Survey, Washington, D. C.; and G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.

Plotting the curves represented by the two equations, they are seen to intersect at (0, 1) and also in the neighborhood of (.6, .8).

$$\frac{\log(7-6^y)}{\log 5}-\frac{\log(4-3^y)}{\log 2}=0.$$

By Double Position the roots near .6, .8 are found to be $x=.565585$, $y=.841307$; these with the values $x=0$, $y=1$ seem to be the only real roots.

Also solved by *JOHN G. KELLER*, State Normal School, Albany, N. Y.
The Proposer sent results only, to-wit: $x=.56556312$, $y=.8413092$.

GEOMETRY.

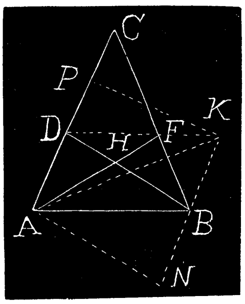
123, Proposed by P. C. CULLEN, Indianola, Iowa.

If the bisectors of the base angles of a triangle are equal the triangle is isosceles.

I. Direct Demonstration by G. I. HOPKINS, A. M., Professor of Physics and Astronomy, High School, Manchester, N. H.

CONSTRUCTION. Make $\angle BDK=\angle FAB$, and draw through B the line NK, making $\angle DBK=\angle AFB$. Draw the perpendiculars KP and AN, and join A and K.

DEMONSTRATION. Since angles FAB and AFB are two angles of a triangle, DK and NK will meet.



$\therefore$ triangles DBK and AFB are equal.

$\therefore$ $DK=AB$ and $BK=BF$.

$\angle AHB=\angle ADH+\angle DAH=\angle ADH+\angle BDK$.

$\therefore$ $\angle AHB=\angle ADK$. Also $\angle AHB=\angle HBF+\angle HFB=\angle HBA+\angle HBK$.

$\therefore$ $\angle AHB=\angle ABK$. $\therefore$ $\angle ADK=\angle ABK$.

$\therefore$ $\angle KDP=\angle ABN$. $\therefore$ $\triangle DPK=\triangle ABN$.

$\therefore$ $AN=KP$, and $NB=DP$.

$\therefore$ $\triangle APK=\triangle AKN$.

$\therefore$ $AP=NK$, and $\therefore$ $AD=BK$. $\therefore$ $AB=FB$. $\therefore$ $\triangle ADK=\triangle AFB$.

$\therefore$ $\angle ABD=\angle FAB$. $\therefore$ $\angle DAB=\angle ABF$.

$\therefore$ $AC=BC$, and the triangle ABC is isosceles. Q. E. D.

II. Demonstration by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, University of Chicago, Chicago, Ill.

Let d_a be the length of the bisector of the angle A opposite to side a. Expressing the area of the given triangle and the areas of the triangles into which it is divided by the bisector of angle A, we get

$$\tfrac{1}{2}bc\sin A=\tfrac{1}{2}bd_a\sin\tfrac{1}{2}A+\tfrac{1}{2}cd_a\sin\tfrac{1}{2}A.$$

$$\therefore\ d_a=\frac{2bc}{b+c}\cos\tfrac{1}{2}A=\frac{2\sqrt{[s(s-a)bc]}}{b+c},\text{ where } 2s=a+b+c.$$

Hence, if $d_a=d_b$, $\dfrac{2\sqrt{[s(s-a)bc]}}{b+c}=\dfrac{2\sqrt{[s(s-b)ac]}}{a+c}$.

$$\therefore\ 2(a+c)^2(s-a)b=2(b+c)^2(s-b)a.$$

$$\therefore\ (a+c)^2(b+c-a)b-(b+c)^2(a+c-b)a=0.$$

$$\therefore\ (a-b)[c^3+c^2(a+b)+3abc+ab(a+b)]=0.$$

Since the second factor is positive, it does not vanish. Hence $a=b$.

We have finally received an elementary direct proof of this theorem. The proof by Professor Hopkins is, I believe, without a flaw, and is the proof so long sought for by a number of mathematicians, among whom was Isaac Todhunter. This demonstration of Professor Hopkins' was examined by one of the ablest mathematicians in this country, and was pronounced by him to be correct. The trigonometric proof by Dr. Dickson is also flawless. We are glad to publish both of these proofs since the demonstration which we gave in Vol. VII, page 223, has been assailed.

Last January a year ago, we received an indirect proof from Prof. A. Anderson, of the University of North Carolina. Professor Anderson's demonstration is free from error and is substantially the proof given in Dr. Halsted's Synthetic Geometry, page 44, though Professor Anderson's figure is drawn quite differently from the one in Dr. Halsted's Geometry.

We shall be pleased to receive, from our readers, opinions on the above demonstrations.

153. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

If P, P', Q, Q' be the extremities of two chords of a conic section, and both chords pass through the point A, show that the sum of the squares of the reciprocals of AP, AP', AQ, AQ' is constant.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

$$\frac{1}{AP}=\frac{1+e\cos\theta}{l},\quad \frac{1}{AP'}=\frac{1+e\cos(\pi+\theta)}{l}=\frac{1-e\cos\theta}{l}.$$

$$\frac{1}{AQ}=\frac{1+e\cos(\frac{1}{2}\pi+\theta)}{l}=\frac{1-e\sin\theta}{l}.$$

$$\frac{1}{AQ'}=\frac{1+e\cos(3\pi/2+\theta)}{l}=\frac{1+e\sin\theta}{l}.$$

$$\therefore\frac{1}{(AP)^2}+\frac{1}{(AP')^2}+\frac{1}{(AQ)^2}+\frac{1}{(AQ')^2}$$

$$=\frac{(1+e\cos\theta)^2+(1-e\cos\theta)^2+(1-e\sin\theta)^2+(1+e\sin\theta)^2}{l^2}=\frac{2(2+e^2)}{l^2}=\text{a constant.}$$

158. Proposed by JOHN MACNIE, A. M., Professor of Latin, University of North Dakota.

Show by a simple diagram that:

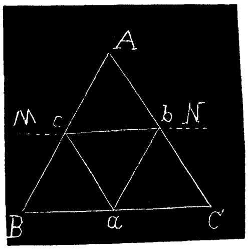
(a) If the angle-sum of an equilateral triangle is constant. that constant is a straight angle.

(b) If the angle-sum is less than a straight angle, the sum increases as the triangle grows less.

(c) If the angle-sum is greater than a straight angle, the sum decreases as the triangle grows less.

Solution by the PROPOSER.

Join the mid-points a, b, c, of the sides of an equilateral triangle ABC. The triangles with vertices A, B, C, being equal, we have $ab=bc=ac$, and $\angle Bac=\angle Cab=\angle Abc$, etc.



I. If $\angle A=\angle bac$, then $\angle A+\angle Abc+\angle Acb=\angle bac+Bac+\angle Cab=180°$, and it is easily shown that $\triangle abc=\triangle Abc$, etc.

Lemma to II and III.

If MN be a line moving towards coincidence with BC, so as always to cut off equal parts on AB, AC; then, according as the angle-sum of ABC is $<$ or $>$ 180°, so will that of $\triangle Abc$ be $<$ or $>$ 180°.

For otherwise, as the angle-sum of Abc would vary from $>$ or $<$ 180° to $<$ or $>$ 180°, there would be some position of MN in which the angle-sum of Abc would be 180°, with consequences incompatible with the hypothesis.

II. If $\angle A<60°$, then $\angle A+\angle Abc+\angle Acb<180°<\angle abc+\angle Abc+\angle Cba$. $\therefore\ \angle abc>\angle A$.

III. If $\angle A>60°$, then $\angle A+\angle Abc+\angle Acb>180°>\angle abc+\angle Abc+\angle Cba$. $\therefore\ \angle abc<\angle A$.

162. Proposed by J. D. PALMER, Providence, Ky.

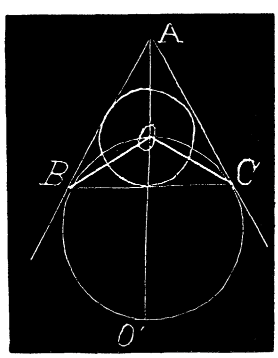
Given the distances from the vertices of a triangle, ABC, to the center of the in-circle, to construct the triangle.

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

AO, BO, $CO=a$, b, c, respectively, where O is the center of the in-circle; BC, AC, $AB=x$, y, z, respectively. Let O_1 be the center of the ex-circle opposite A. Then

$$AO^2=\frac{(p-x)^2}{\cos^2\frac{1}{2}A},\text{ where } p=\tfrac{1}{2}(x+y+z).$$

$$\therefore AO^2=\left(\frac{p-x}{p}\right)yz=yz-\frac{xyz}{p}=yz-4Rr=a^2.$$



$$\text{Similarly, } BO^2=\left(\frac{p-y}{p}\right)xz=xz-4Rr=b^2.$$

$$CO^2=\left(\frac{p-z}{p}\right)xy=xy-4Rr=c^2.$$

$$\therefore AO.BO.CO=abc=\frac{\triangle\, xyz}{p^2}=4Rr^2.$$

$$\therefore 4Rr=abc/r, \text{ and } (a^2r+abc)/r=yz,\ (b^2r+abc)/r=xz,\ (c^2r+abc)/r=xy.$$

$$\therefore x(a^2r+abc)/r=y(b^2r+abc)/r=z(c^2r+abc)/r$$

$$=\sqrt{[(a^2r+abc)(b^2r+abc)(c^2r+abc)/r^3]}.$$

Also $a^2x+b^2y+c^2z=xyz\left(\frac{p-x}{p}+\frac{p-y}{p}+\frac{p-z}{p}\right)=xyz=1.$

$$\therefore \frac{a^2}{yz}+\frac{b^2}{xz}+\frac{c^2}{xy}=1, \text{ or } \frac{ar}{ar+bc}+\frac{br}{br+ac}+\frac{cr}{cr+ab}.$$

$$\therefore 2abcr^2+(a^2b^2+a^2c^2+b^2c^2)r^2=a^2b^2c^2.$$

$$\therefore r^3+Ar^2=B. \text{ Let } r=S-\tfrac{1}{3}A.$$

$$\therefore S^3-\tfrac{1}{3}A^2S=B-2A^3/27.$$

$$\therefore S=\left(\tfrac{1}{2}B-\tfrac{1}{27}A^3+\sqrt{\frac{B^2}{4}-\frac{A^3B}{27}}\right)^{\frac{1}{3}}+\left(\tfrac{1}{2}B-\tfrac{1}{27}A^3-\sqrt{\frac{B^2}{4}-\frac{A^3B}{27}}\right)^{\frac{1}{3}}.$$

This determines r and therefore x, y, z, the sides of the triangle.

Otherwise, draw AO and produce AO to O_1 so that $OO_1=bc/r$. Upon OO_1 as diameter describe a circle. With O as center and b as a radius describe an arc cutting the circle in B. Similarly, with O as center and c as radius, draw an arc cutting the circle in C. Join BC, AC, AB, then ABC is the triangle required. For O_1 is the ex-center opposite A by construction as follows:

$$AO.AO_1=yz=(AO^2r+AO.BO.CO)/r.$$

$$\therefore AO_1=(AOr+BO.CO)/r.$$

The solution published in the last issue contained an error in the fourth line, and this vitiated the whole solution. ED.

163. Proposed by J. C. NAGLE, Professor of Civil Engineering in the Agricultural and Mechanical College of Texas, College Station, Tex.

Given the equal sides of an isosceles triangle and the radius of the inscribed circle to solve the triangle. As a numerical example let the known sides be 27 and the radius of the inscribed circle 3.5. The problem occured in connection with some mill work and the exterior angles of the triangle were required in order to make patterns for iron braces.

I. Solution by A. H. HOLMES, Professor of Mathematics, Bowdoin College, Brunswick, Me.

Let ABC be the triangle, O the center of the inscribed circle, and OD the perpendicular from O to the side AB, its radius; let $\angle BOD=\theta$ and $\angle AOD=\phi$. Then we have $\tan\theta+\tan\phi=\frac{54}{7}$. But $\phi+2\theta=180°$.

$$\therefore \tan\phi=-\tan 2\theta=-\frac{2\tan\theta}{1-\tan^2\theta}.$$

$\therefore \tan^3\theta-\frac{54}{7}\tan^2\theta+\tan\theta+\frac{54}{7}=0.$

From this equation we find $\tan\theta=1.16425$.

$\therefore \angle ABC=\angle ACB=81°\ 19'\ 14''$, and $\angle BAC=17°\ 21'\ 32''$.

We now readily find $BC=8.149+$.

II. Solution by H. C. WHITAKER, C. E., Ph. D., Manual Training School, Philadelphia, Pa.; G.B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; and L. C. WALKER, A. M., Petaluma High School, Petaluma, Cal.

$$r=(s-a)\tan\tfrac{1}{2}A=\tfrac{1}{2}(b+c-a)\sqrt{\frac{(a+c-b)(a+b-c)}{(a+b+c)(b+c-a)}}.$$

But $b=c$. $\therefore r=a\sqrt{\dfrac{2b-a}{2b+a}}$.

$\therefore a^3-2a^2b+4ar^2+8br^2=0.$

This cubic can be solved, and gives the values of the unknown sides.

Substituting values, we get $a^3-54a^2+49a+2646=0$.

$\therefore a=8.1498$, 52.0838, or -6.2336. The first and second values are both admissible. $\cos B=a/2b=.15092$ or $.96451$.

$\therefore B=81°\ 19'$, or $15°\ 18'$.

Also solved by *J. SCHEFFER*.

We received a solution of problem 156 from Prof. Henry Heaton too late for credit in the number in which the solution of the problem was published. Professor Heaton gave construction also when the point does not lie between the two parallels. As the construction is similar to the case when the point lies between the two parallels we will not publish the construction.

CALCULUS.

REMARKS ON THE SOLUTION OF PROBLEM 123.

By Wm. E. Heal, Marion, Ind.

A remarkable error occurs in the published solution of Problem 123, in writing $1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\ldots=\frac{\pi^3}{35}$.

The summation referred to by Professor Echols and the Editor is incorrectly quoted. The true formula is

$$\tau_{2m+1}=\frac{E_m\pi^{2m+1}}{2^{2(m+1)}(2m)!}=1-\frac{1}{3^{2m+1}}+\frac{1}{5^{2m+1}}-\ldots\ldots$$

See Chrystal's *Algebra*, Part II, page 342. For $m=1$ this becomes, since $E_1=1$,

REMARK. On again looking up our reference we find that we quoted the reference correctly, though as Mr. Heal observes, it seems that our reference is wrong. The value of the series

$$1-\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}-\frac{1}{7^{2n+1}}+\text{etc.}=\frac{\pi^3}{32}$$

is also given in Carr's *Synapsis of Pure Mathematics*, page 432. Since it now appears that the *value* of the integral is not known in finite terms, we will renew our offer to give the person finding such value a year's subscription to the MONTHLY. ED.

125. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Show that the *complete primitive* of the differential equation

$$\left[\tan^{-1}(x)-\frac{x}{1+x^2}\right]\frac{dy^2}{dx^2}=1\left[\frac{x}{(1+x^2)^2}\right]\left[x\frac{dy}{dx}-y\right]$$

is $y=C\tan^{-1}(x)+cx$.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

w being one solution of

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0\ldots\ldots(1),$$

and P, Q functions of x, $y=vw\ldots\ldots(2)$ is given, by the usual theory, from

$$\frac{d^2v}{dx^2}+\left(\frac{2}{w}\frac{dw}{dx}+P\right)\frac{dv}{dx}=0\ldots\ldots(3),\text{ or } v=c_2+c_1\int\frac{dx}{w^2}e^{-\int Pdx}\ldots\ldots(4).$$

In the given equation, $w=x\ldots\ldots(5)$, and

$$P=-\frac{2x^2}{(1+x^2)^2}\div\left[\tan^{-1}x-\frac{x}{1+x^2}\right]\ldots\ldots(6).$$

(4) now easily becomes

$$v=c_2+c_1\int\frac{dx}{x^2}\left(\tan^{-1}x-\frac{x}{1+x^2}\right)=c_2+c_1\left(\int\frac{\tan^{-1}x}{x^2}-\int\frac{dx}{x(1+x^2)}\right)$$

$$=c_2-c_1\frac{\tan^{-1}x}{x^2}\ldots\ldots(7),$$

and (2) is $y=c_2x-c_1\tan^{-1}x\ldots\ldots(8)$.

II. Solution by W. W. BEMAN, A. M., Professor of Mathematics, State University, Ann Arbor, Mich.

$$\left(\tan^{-1}x-\frac{x}{1+x^2}\right)\frac{d^2y}{dx^2}=\frac{2x}{(1+x^2)^2}\left(x\frac{dy}{dx}-y\right).$$

Writing the equation in the form

$$\frac{d}{dx}\left(\frac{\tan^2 x}{x}\right)\cdot\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{1+x^2}\right)\cdot\frac{d}{dx}\left(\frac{y}{x}\right)$$

it is obvious that $y=\tan^{-1}x$ and $y=x$ are independent particular integrals. Hence the complete primitive is $y=c_1\tan^{-1}x+c_2x$.

Solutions of this problem were also received from *L. C. WALKER*, and *G. B. M. ZERR*.

126. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Find the volume contained between the conical surface whose equation is $z=a-\surd(x^2+y^2)$, and the planes whose equations are $x=z$ and $x=0$ by the formula $\int\int\int dxdydz$. [Todhunter's *Integral Calculus.*]

Solution by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

The cone clearly extends from vertex $(0, 0, a)$ towards $z=0$. Hence in

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} dxdydz$$

we have $z_1=x$; $z_2=a-\surd(x^2-y^2)$; $x=a-\surd(x^2+y^2)$.
$\therefore$ $y^2=a^2-2ax$, $x_1=0$, and for x_2 we have $y_1=y_2$. $\therefore$ $x=\frac{1}{2}a$.

$$\therefore \text{ Volume}=\int_0^{\frac{1}{2}a}\int_{y_1}^{y_2}[a-x-\surd(x^2+y^2)]dxdy$$

$$=\int_0^{\frac{1}{2}a}\left[(a-x)(y_2-y_1)-\frac{y}{2}\surd(x^2+y^2)-\frac{x^2}{2}\log y+\surd(x^2+y^2)\right]_{y_1}^{y_2}dx$$

$$=\int_0^{\frac{1}{2}a}\left[2(a-x)\surd(a^2-2xa)-\surd(a^2-2xa)(a-x)-\frac{x^2}{2}\log\frac{a-x+\surd(a^2-2ax)}{a-x-\surd(a^2-2ax)}\right]dx.$$

Put $2x=a\sin^2\varphi$.

$$\therefore\ V=\int_0^{\frac{1}{2}\pi}a\sin\phi\cos\phi d\phi\left[(1+\cos\phi)\frac{a^2}{2}\cos\phi-\frac{a^2}{8}\sin^4\phi\log\left(\frac{1+\cos\phi}{1-\cos\phi}\right)^2\right]$$

$$=\frac{a^3}{2}\int_0^{\frac{1}{2}\pi}\left[\sin\phi\cos^2\phi+\sin\phi\cos^4\phi-\sin^5\phi\cos\phi\log\frac{1+\cos\phi}{\sin\phi}\right]d\phi$$

$$=\frac{a^3}{2}(\tfrac{1}{3}+\tfrac{1}{5})-\left[\frac{a^3\sin^6\phi}{12}\log\frac{1+\cos\phi}{\sin\phi}\right]_0^{\frac{1}{2}\pi}-\frac{a^3}{12}\int_0^{\frac{1}{2}\pi}\sin^6\phi\left(\frac{\cos\phi}{\sin\phi}+\frac{\sin\phi}{1+\cos\phi}\right)dx$$

$$=\tfrac{4}{15}a^3-\tfrac{1}{12}a^3\int_0^{\frac{1}{2}\pi}[\cos\phi\sin^5\phi+\sin^5\phi(1-\cos\phi)]d\phi=\tfrac{4}{15}a^3-\frac{a^3}{12}\cdot\frac{4.2}{5.3}=\tfrac{2}{9}a^3.$$

Similar solutions were received from *GEORGE LILLEY, LON C. WALKER, G. B. M. ZERR*, and *J. SCHEFFER*.

MECHANICS.

127. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Develop the Fourier Series to represent the temperature of a circular wire of uniform cross-section, in which the temperatures of the four quadrants are in order $t, 2t, 3t, 4t$.

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

From Fourier's *Analytical Theory of Heat*, page 217, we get for the complete solution of the linear and varied movement of heat in a ring after a time T the following:

$$v=e^{-hT}\left[\frac{1}{2\pi}\int_0^{2\pi}f(x)dx+\frac{1}{\pi}\sum_{n=1}^{n=\infty}e^{-n^2kT}\sin nx\int_0^{\frac{1}{2}\pi}\sin nxf(x)dx\right.$$

$$\left.+\frac{1}{\pi}\sum_{n=1}^{n=\infty}e^{-n^2kT}\cos nx\int_0^{\frac{1}{2}\pi}\cos nxf(x)dx\right],$$

$$\text{where } f(x)=\frac{1}{2\pi}\int_0^{2\pi}f(x)dx+\frac{1}{\pi}\sum_{n=1}^{n=\infty}\sin nx\int_0^{2\pi}\sin nxf(x)dx$$

$$+\frac{1}{\pi}\sum_{n=1}^{n=\infty}\cos nx\int_0^{2\pi}\cos nxf(x)dx.$$

$$\int_0^{2\pi}f(x)dx=t\int_0^{\frac{1}{2}\pi}dx+2t\int_{\frac{1}{2}\pi}^{\pi}dx+3t\int_{\pi}^{\frac{3}{2}\pi}dx+4t\int_{\frac{3}{2}\pi}^{2\pi}dx=5\pi t.$$

$$\int_0^{2\pi}\sin nxf(x)dx=t\int_0^{\frac{1}{2}\pi}\sin nxdx+2t\int_{\frac{1}{2}\pi}^{\pi}\sin nxdx+3t\int_{\pi}^{\frac{3}{2}\pi}\sin nxdx+4t\int_{\frac{3}{2}\pi}^{2\pi}\sin nxdx$$

$$=\frac{t}{n}(\cos\tfrac{1}{2}\pi n+\cos\pi n+\cos\tfrac{3}{2}\pi n)-\frac{3t}{n}=-\frac{4t}{n},\text{ except when } n=4m.$$

$$\int_0^{2\pi}\cos nxfxdx=t\int_0^{\frac{1}{2}\pi}\cos nxdx+2t\int_{\frac{1}{2}\pi}^{\pi}\cos nxdx+3t\int_0^{\frac{3}{2}\pi}\cos nxdx+4t\int_{\frac{3}{2}\pi}^{2\pi}\cos nxdx$$

$$=-\frac{t}{n}(\sin\tfrac{1}{2}\pi n+\sin\tfrac{3}{2}\pi n)=0.$$

$$\therefore f(x)=\tfrac{5}{2}t-(t/\pi)(4\sin x+2\sin 2x+\tfrac{4}{3}\sin 3x+\tfrac{4}{5}\sin 5x+\tfrac{4}{6}\sin 6x+\tfrac{4}{7}\sin 7x+\tfrac{4}{9}\sin 9x+\ldots)$$

$$\therefore\ V=e^{-hT}[\tfrac{5}{2}t-(t/\pi)(4\sin xe^{-kT}+2\sin 2xe^{-2^2kT}+\tfrac{4}{3}\sin 3xe^{-3^2kT}+\tfrac{4}{5}\sin 5xe^{-5^2kT}+\ldots)]$$

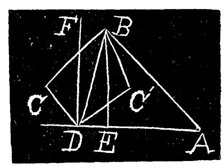
AVERAGE AND PROBABILITY.

108. Proposed by A. H. HOLMES, Brunswick, Me.

Required the average area of the quadrilateral whose sides are a, b, c, and d.

I. Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $ABCD$ be the quadrilateral, $AB=a$, $AD=b$, $BC=c$, $CD=d$, $a>b>c>d$, $BE=v$, $DE=u$,.



When the quadrilateral is convex, the average area is, (the average area of ABD) + (the average area of BCD).

When the quadrilateral is concave, the average area is, (the average area of ABD) − (average area of $BC'D$).

Since BCD is equal to $BC'D$, the average area required is the average area of $ABD=\triangle$. Area $ABD=\frac{1}{2}ab\sin A$.

Now $DB=\sqrt{(u^2+v^2)}$.

$\therefore\ \cos A=(a^2+b^2-u^2-v^2)/2ab$.

$\therefore$ Area $ABD=\frac{1}{4}\sqrt{[4a^2b^2-(a^2+b^2-u^2-v^2)^2]}$.

But $v^2=a^2-(b-u)^2$.

$\therefore$ Area $ABD=\frac{1}{2}b\sqrt{[a^2-(b-u)^2]}=C$.

The limits of u are 0 and $[(c+d)^2+b^2-a^2]/2b=u'$.

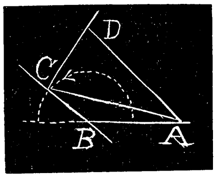
$$\therefore\triangle=\frac{\int_0^{u'}Cdu}{\int_0^{u'}du}=\frac{b}{2u'}\int_0^{u'}\sqrt{[a^2-(b-u)^2]}du.$$

$$\therefore\triangle=\frac{b^2}{(c+d)^2+b^2-a^2}\Bigg[\frac{b}{2}\sqrt{(a^2-b^2)}+\frac{a^2}{2}\sin^{-1}\frac{b}{a}-\frac{a^2+b^2-(c+d)^2}{8b^2}$$

$$\sqrt{\{4a^2b^2-[a^2+b^2-(c+d)^2]^2\}}-\frac{a^2}{2}\sin^{-1}\left(\frac{a^2+b^2-(c+d)^2}{2ab}\right)\Bigg].$$

II. Solution by F. P. MATZ. Sc. D., Ph. D., Professor of Mathematics and Astronomy, Defiance College, Defiance. O.

Let $ABCD$ be the quadrilateral, side $AB=a$, $BC=b$, $CD=c$, $DA=d$, $\angle ABC=\theta$, and $\angle CDA=\phi$.



Suppose the vertices of the quadrilateral to be movable, or hinged; then the side BC may make a complete revolution about B as a center so long as $(c+d)$ is not less than $(a+b)$. That is, for $(c+d)$ not less than $(a+b)$, the angle θ will vary uniformly from 0° to 360°; but as soon as the side BC has moved below the side AB produced, we have hour-glass quadrilaterals composed of two practically isolated triangles which are not to be considered in finding the average area of the quadrilateral (a, b, c, d). We are, therefore, constrained to regard θ as varying uniformly from 0° to 180°; but if $(c+d)<(a+b)$, which becomes certain when the sides of the quadrilateral (a, b, c, d) are numerically expressed, we are constrained to regard θ as varying uniformly from 0° to $\cos^{-1}\{[a^2+b^2-(c+d)^2]/2ab\}$.

From the diagram it is evident that $Q=(\triangle ABC+\triangle CDA)=\frac{1}{2}(ab\sin\theta+cd\sin\phi)$. Now $AC=\sqrt{(a^2+b^2-2ab\cos\theta)}$, and

$$\cos\phi=\frac{c^2+d^2-(a^2+b^2-2ab\cos\theta)}{2cd}.$$

[For the sake of brevity, put $m=c^2+d^2-a^2-b^2$, $n=4c^2d^2-m^2$, $p=n/4a^2b^2$, and $q=m/ab$].

$$\therefore\ \sin\phi=\sqrt{1-\left(\frac{m+2ab\cos\theta}{2cd}\right)^2}=\sqrt{\frac{(4c^2d^2-m^2)-4abm\cos\theta-4a^2b^2\cos^2\theta}{4c^2d^2}}$$

$$=\frac{1}{2cd}\sqrt{(n-4abm\cos\theta-4a^2b^2\cos^2\theta)}=(ab/cd)\sqrt{(p-q\cos\theta-\cos^2\theta)}$$

$$=(ab/cd)\sqrt{[(p+\tfrac{1}{4}q^2)-(\tfrac{1}{2}q+\cos\theta)^2]}.$$

Also, $Q=\frac{1}{2}ab\{\sin\theta+\sqrt{[(p+\frac{1}{4}q^2)-(\frac{1}{2}q+\cos\theta)^2]}\}$.

Representing $\cos^{-1}\{[a^2+b^2-(c+d)^2]/2ab\}$ by θ_1, the expression for the average area of the quadrilateral on the hypothesis that the interior angle at B vary uniformly from 0 to θ_1 becomes

$$Q_B=\frac{ab}{2\theta_1}\int_0^{\theta_1}\{\sin\theta+\sqrt{[(p+\tfrac{1}{4}q^2)-(\tfrac{1}{2}q+\cos\theta)^2]}\}d\theta.$$

Similar operations give Q_C, Q_D, and Q_A; and, therefore, the required average area of the quadrilateral (a, b, c, d) becomes $\mathbf{Q}=\frac{1}{4}(\mathbf{Q}_A+\mathbf{Q}_B+\mathbf{Q}_C+\mathbf{Q}_D)$. Slight modifications, in signs, etc., may be occasioned by quadrilaterals having a re-entrant angle.

MISCELLANEOUS.

104. Proposed by HARRY S. VANDIVER, Bala, Pa.

A Theorem of Fermat. The area of a right angled triangle with commensurable sides cannot be a square number. [Cf. Chrystal's *Algebra*, Vol. II., page 535.]

Solution by L. C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Since the sides of the right-angled triangle may have a common factor, we let af, bf, and cf represent its sides, where a, b, c are prime to each other. Then we have $a^2+b^2=c^2$......(1).

For the area, $\frac{1}{2}abf^2$, of the triangle to be a square, $\frac{1}{2}ab$ must be a square.

In (1) either a or b must be even and the other odd, because the sum of two odd squares can not be a square. Assume a even and b odd. Then for $\frac{1}{2}ab$ to be a square, b must be a square, and a must be double a square.

Let $a=2m^2$, $b=n^2$, and substitute these values in (1); then we have

$$(2m^2)^2+(n^2)^2=c^2\text{......}(2).$$

Now set $2m^2=2pq$, and $n^2=p^2-q^2$; or

$$n^2+q^2=p^2\text{......}(3).$$

In (3) p is odd, and we will assume q even. Let $q=2\alpha\beta$, $p=\alpha^2+\beta^2$, and we find $m=2\alpha\beta(\alpha^2+\beta^2)$......(4).

For $2\alpha\beta$ to be a square, we will assume $\alpha=2m_1^2$ and $\beta=n_1^2$. Substituting these values in (4), we obtain

$$m^2=4m_1^2n_1^2(4m_1^4+n_1^4)\text{......}(5).$$

In order that the right member of (5) may be a square, we must have

$$4m_1^4+n_1^4=c_1^2,$$

say; which is of the same form of (2). But c_1^2 is less than c^2. Proceeding in exactly the same way, we can reduce c^2 indefinitely. By our hypothesis c^2 can not be reduced to zero nor less. Hence, results *Fermat's Theorem.*

PROBLEMS FOR SOLUTION.

ARITHMETIC.

154. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Suppose there is a meadow of 8 acres in which the grass grows uniformly, and that 21 oxen could eat up the whole pasture in 6 weeks, or 18 oxen in 9 weeks; what number of oxen diminished by the removal of 9, at the end of 14 weeks, could eat it up in 18 weeks ?

155. Proposed by F. P. MATZ. Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

A bought a horse, which he sold to B at a *loss* of $m=6\%$; B sold the horse to C at a *loss* of $n=5\%$; and C sold the horse to D at a *gain* of $p=12\frac{1}{2}\%$. How much did A *lose*, if C gained $\$G=\$26.79$?

ALGEBRA.

156. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

$(z+x)a-(z-x)b=2yz$......(1); $(x+y)b-(x-y)c=2xz$......(2); $(y+z)c-(y-z)a=2xy$......(3). Find the values of x, y, and z, by the method of linear simultaneous equations.

NOTE. This problem was somewhat abbreviated in the last issue. The problem occurs in Fisher & Schwatt's *Elements of Algebra*, page 224, under Linear Simultaneous Equations.

157. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Solve the equations,

$$\frac{x}{a+\lambda}+\frac{y}{b+\lambda}+\frac{z}{c+\lambda}+\frac{u}{d+\lambda}=1, \qquad \frac{x}{a+\mu}+\frac{y}{b+\mu}+\frac{z}{c+\mu}+\frac{u}{d+\mu}=1,$$

$$\frac{x}{a+\nu}+\frac{y}{b+\nu}+\frac{z}{c+\nu}+\frac{u}{d+\nu}=1, \qquad \frac{x}{a+\rho}+\frac{y}{b+\rho}+\frac{z}{c+\rho}+\frac{u}{d+\rho}=1.$$

158. Proposed by R. D. BOHANNON, Ph. D., Professor of Mathematics, Ohio State University, Columbus, O.

If $\frac{x}{a+\alpha}+\frac{y}{b+\alpha}+\frac{z}{c+\alpha}=\frac{x}{a+\beta}+\frac{y}{b+\beta}+\frac{z}{c+\beta}=\frac{x}{a+\gamma}+\frac{y}{b+\gamma}+\frac{z}{c+\gamma}=1,$

show, $\frac{x}{(a+\beta)^2}+\frac{y}{(b+\beta)^2}+\frac{z}{(c+\beta)^2}=\frac{(\gamma-\beta)(\alpha-\beta)}{(a+\beta)(b+\beta)(c+\beta)}.$

GEOMETRY.

182. Proposed by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

Show how to cut from a given cube, edge s, the maximum tetrahedron.

183. Proposed by S. F. NORRIS, Professor of Mathematics and Astronomy, Baltimore City College, Baltimore, Md.

"Two quadrilaterals having three sides of the one equal to the three corresponding sides of the other, each to each, and the two corresponding angles adjacent to the unknown sides equal, each to each, are equal figures." [Olney's *Geometry*, page 129.]

CALCULUS.

148. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Hemholtz's differential equation for the strength of an electric current C at any time t, is $C=E/R-L/R\times dC/dt$. Solve this equation, supposing $C=0$ when $t=0$; and E, R, L are to be regarded as constants.

149. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Find the volume contained between the plane $z=(a-x)\cot\beta$ and the surface $xz^2=(a-x)(x^2+y^2)$.

MECHANICS.

138. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

A smooth elliptical tube is held in the vertical plane with its major axis inclined to the vertical. A particle is projected from the lowest point. Find the pressure on the tube at any point and the condition that the pressure may vanish at the highest point.

139. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

A homogeneous sphere, radius $r=50$ inches, makes $m=30$ revolutions around an axis every second. The mass begins to disappear from the surface into space at a rate exactly sufficient to cause the diameter to decrease uniformly at the rate of $(1/n)$th$=1/1000$th of a linear inch per second. At what rate per second is the angular velocity of the sphere changing the instant the diameter becomes $p=10$ inches? What is the diameter of the sphere when the rate of disappearance of matter is midway between minimum and maximum? When is the angular velocity a maximum? How does this maximum angular velocity compare with the original angular velocity? What is the diameter of the sphere when the paracentric force is (1) a maximum and (2) a minimum?

DIOPHANTINE ANALYSIS.

101. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

If p and q are such values of x and y as fulfill the conditions $x^2\pm y^2-1=$a square, find, in terms of p and q, the expression for an indefinite number of other values.

102. Proposed by A. H. BELL, Hillsboro, Ill.

Prove that every indeterminate equation of the second degree can be reduced to $x^2-Ay^2=Bz^2$. [*Legendre.*]

AVERAGE AND PROBABILITY.

123. Proposed by LON C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Three points are taken at random within a square. What is the probability that the triangle formed by joining them is acute?

124. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Find the average area of a spherical polygon of $n=6$ sides.

MISCELLANEOUS.

124. Proposed by J. W. YOUNG, Graduate Student. Cornell University, Ithaca. N. Y.

Prove that the general value of θ, which satisfies the equation

$$(\cos\theta+i\sin\theta)(\cos2\theta+i\sin2\theta)\ldots\text{to } n \text{ factors}=1 \text{ is } \frac{4m\pi}{n(n+1)};$$

where m is any integer $(i=\sqrt{-1})$.

125. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Assume $m=nt+\varepsilon-\omega$, thus giving $v=m+e\sin v$ as the relation connecting the mean and eccentric anomalies, then express $x=a\cos v$, $y=b\sin v$, and $r=a(1-e\cos v)$ by a Fourier series in terms of m.

BOOKS AND PERIODICALS.

Graphs. By Robert J. Aley, A. M., Ph. D., Professor of Mathematics in the Indiana University. Pamphlet Form, 21 pages. Price, 10 cents. Boston and Chicago: D. C. Heath & Co.

This is No. 6 of Heath's Mathematical Monographs issued under the general editorship of Webster Wells, S. B., Professor of Mathematics in the Massachusetts Institute of Technology. In this monograph, Dr. Aley has pointed out the various uses that are being made of Graphs in almost every department of knowledge. In addition to plotting a temperature curve, four problems are solved by means of graphs, of which the first is the following: "A travels 4 miles an hour, B 6 miles an hour. If A has two hours the start, when and where will B overtake him?" Then comes the solution of Linear Equations in two variables, and following these, Simultaneous Quadratics, closing with the graphic representation of complex numbers.

College Algebra. By Leonard Eugene Dickson, Ph. D., Assistant Professor of Mathematics in the University of Chicago. First edition. First thousand. 8 vo. Cloth, vi+214 pages. Price, $1.50. New York: John Wiley & Sons.

"This text," the author tells us, "is intended primarily for college and technical schools. By treating only the subjects usually given in the college course in algebra, space has been gained for more detailed exposition of the more difficult topics." The work begins with a treatment of Number in Algebra; Surds and Imaginaries. Then follows in order the subjects Exponents—Logarithms; FactorTheorem—QuadraticEquations; Simultaneous Equations—Determinants; Ratio—Proportion, Variation, the Progressions, Compound Interest and Annuities; Undetermined Coefficients, Partial Fractions; Permutation and Combination, Binomial and Multinomial Theorem; Probability, Mathematical Induction, Limits, Indeterminate Forms; Convergency and Divergency of Series; Power Series and other series, Summation of Series, the Method of Differences, Graphic Algebra,

Theory of Equations and an Appendix. Every subject treated by Dr. Dickson is treated in a thoroughly scientific manner, and with that accuracy and rigor so characteristic of all of his mathematical work. Dr. Dickson deserves the highest praise for sacrificing his time so fully devoted to original investigations in the various branches of mathematics and giving it to the preparation of text-books to be used in colleges. We hope that this edition will soon be exhausted, and that in the second edition he will add considerable material on the still more elementary parts of algebra, thus adapting the work to a larger constituency.

The Public School Arithmetic. For Grammar Grades, based on McLellan and Dewey's "Psychology of Numbers." By J. A. McLellan, A. M., LL. D., and A. F. Ames, A. B. 16 mo. Cloth. xii+369 pages. Price, 60 cents. New York: The Macmillan Co.

This book is based on Dewey & McLellan's "Psychology of Numbers," the basal idea of which is the correlation of number and measurement. They hold, *no number without measurement, no measurement without number*. This idea of number has been, I think, very successfully refuted a number of times by a number of different mathematicians and educators. Certainly the idea of number existed a long time before the idea of measurement. If one were to count the number of stone in a stone fence, the unit of measure in this case is one certainly very hard to adjust to our modern notion of a unit of measure. It would seem that a book, based upon what seems to a great number, perhaps the greater number, of our best educators to be erroneous, would have very little to commend it. But this is not true in regard to the book under consideration. The problems for solution are thoroughly practical, being taken from every source of modern life, commercial, agricultural, industrial and social. These problems are presented in such a way that the pupil can easily realize the conditions which gave rise to them. The systematic arrangement of the solutions of illustrative problems is another feature of the work which is highly commendable. These solutions are given in steps, though steps are not numbered. The subject of mensuration which, because of pressure from outside, has been omitted entirely by a number of authors, has received attention in this book commensurate with its importance. There are a number of other features that are commendable, for the appreciation of which the book should be consulted.

The American Journal of Mathematics. Edited by Frank Morley with the cooperation of Simon Newcomb and other mathematicians. Published under the auspices of the Johns Hopkins University. Price, $5.00 per year in advance. No. 1, Vol. xxiv, contains to following articles:

Cyclic Group of the Simple Ternary Linear Fractional Group in a Galois Field, by L. E. Dickson; Curves of Triple Curvature, by James G. Hardy; Primary Prime Functions in Several Variables and a Generalization of an Important Theorem of Dedekind, by Harris Hancock; On Certain Properties of the Plane Cubic Curve in Relation to Circular Points at Infinity, by R. A. Roberts; Estimates of Pierce's Linear Associative Algebra, by H. E. Hawkes; Groups Defined by the Orders of Two Generators and the Order of Their Product, by G. A. Miller.

Annals of Mathematics. Edited by Ormond Stone, W. E. Byerly, W. F. Osgood and others. Published quarterly under the auspices of Harvard University. Price, $2.00 per year. No. 2, Vol. 3, second series, January, 1902, contains the following articles:

Some Applications of the Method of Abridged Notation, by Professor Maxime Bocher; On Roots of Functions Connected by a Linear Recurrent Relation of the Second Order, by Dr. M. B. Porter; Space of Constant Curvature, by Professor F. S. Woods.

Cyclographic Transformation of Ordinary Space. By Dr. Arnold Emch. Reprint from the University of Colorado Studies.

The following periodicals have been received: *The Open Court*, for February; *Educational Times*, for February; *Kansas University Quarterly; Bulletin of the American Mathematical Society; Scientific American;* and *The American Monthly Review of Reviews.*

EUGENIO BELTRAMI.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. IX. MARCH, 1902. NO. 3.

EUGENIO BELTRAMI.

By DR. GEORGE BRUCE HALSTED.

The great vindicator and interpreter of non-Euclidean geometry, Beltrami, like its two creators Bolyai and Lobachevski, was noted as a student for his insubordination.

Beltrami was born at Cremona on November 16th, 1835, and attended the elementary schools, gymnasium and liceum of that city, except for the scholastic year 1848-49 in which he was at that gymnasium of Venice which now bears the name of Marco Polo.

Having finished the liceal studies in the summer of 1853, in the following November he inscribed himself as student in the Mathematical Faculty of the University of Pavia, after having obtained there a place on the Castiglioni Foundation in the Collegio Ghislieri.

But in the succeeding year, accused of having promoted disorders against the Abbot Leonardi, rector of this college, he was expelled from it, together with five others of his college mates. Thus, like Lobachevski, not even being a charity beneficiary could restrain his irrepressible independence.

This expulsion worked a terrible hardship and disadpointment on the ambitious youth. As Loria says in his notice of Beltrami, on which we draw here, "This measure—perhaps not absolutely without grounds, but certainly too rigorous—had disastrous consequences for its victim."

For, though the University was still open to him, and though he had the

great good fortune to attend certain lectures of Brioschi, yet the poverty of his family constrained him to return home before having passed the examination which precedes the doctorate. In November, 1856, he went to Verona where he had obtained the humble position of secretary to the engineer Diday in the government service of Lombardy-Venice. Here he remained until the 10th of January, 1857, when *for political reasons* he was bruskly dismissed by the director-general, Busche.

Fortunately the annexation of Lombardy to Piedmont, happening soon after, allowed Diday to transfer his office to Milan, taking his secretary with him.

At Milan Beltrami undertook all over again his mathematical education. Here he was so fortunate as to have access again to his former professor, Brioschi, and also to Luigi Cremona.

The friendship of these great men was of decisive influence for his life, opening to him at a stroke the very career for which he had longed. Thus he who for lack of a degree had seen himself disbarred from the secondary schools and from the corps of military engineers, was, on the basis of his publications in the *Annali di Matematica*, named (Oct. 18th, 1862,) "Professore straordinario" in the University of Bologna.

It is to the honor of Cremona to have suggested this appointment, and of Brioschi, then secretary-general to the Minister of Public Instruction, to have adopted the suggestion. By it Beltrami was at one stroke liberated from the shackles of a humble administrative occupation and put in position to consecrate himself wholly to the genial occupation for which he was by nature so pre-eminently fitted.

The very next year, on the proposal of Enrico Betti, he was offered the professorship of geodesy in the University of Pisa. From Pisa he returned to Bologna in September, 1866, as professor of rational mechanics.

But it was ideas of a geodetic character which gave that turn to his creative genius destined to be crowned with such brilliant fame. In the exordium of a memoir dated Pisa, May 31st, 1866, Beltrami remarks that in treating of a map destined to serve for measurements of distance it would be most convenient to determine, that to the geodetics of the surface should correspond the straights of the plane, because, such a representation accomplished, the questions concerning geodetic triangles would be reduced to simple questions of plane trigonometry. He concludes that "the only surfaces capable of being represented on a plane so that to every point corresponds a point and to every geodetic a a straight are those whose curvature is everywhere constant."

Now at the very time that Beltrami was working on surfaces of constant curvature, Baltzer in Germany, Hoüel in France and Battaglini in Italy had set to work to diffuse the revolutionary ideas of Bolyai and Lobachevski, while Dedekind published the "Habilitationsvorlesung" of Riemann: "On the hypotheses which lie at the basis of geometry."

Of this Beltrami made an annotated translation of which he speaks in two letters to Genocchi: "Bologna, 9 June, 1868. * * * The manuscript of

the translation of a posthumous memoir of Riemann, most interesting for the extreme importance and vastness of its subject, to which, it is necessary to say, the brevity of the development is little adequate.

As to my translation I ought to mention in the first place that I have tried to be most faithful to the text, for fear that doing otherwise I might alter the true sense of a composition which in many points gives place for doubts of interpretation.

At the end of the translation I have inserted some annotations, made *currenti calamo* and for *internal use* (employing the phrase of the apothecaries), that is to say destined properly for myself alone. Hence such notes either should not be read, or not without the greatest indulgence, because most of them are rather indications destined to serve me as guides to call attention to those points that occasioned me the most difficulty, than comments properly so called and definitive.

There are then very many other points which merit elucidation. You will find in two of my annotations certain allusions to a mode of mine of interpreting the results of the non-Euclidean geometry. If you wish any hint about this you might turn to Professor Cremona, who has had in his hands a manuscript of mine containing the complete *real* construction of the planimetry of Lobachevski. The ground of the researches of Lobachevski lies beyond doubt in the doctrine touched by Riemann. I intend to publish some of my studies on this subject, encouraged by the support which they find in the work of Riemann, only lately come to my cognizance.''

''Bologna, 23 July, 1868.

The past year, when no one knew of this fundamental work of Riemann, I had communicated to Cremona a paper of mine in which I gave an interpretation of the non-Euclidean planimetry, which seemed to me satisfactory. Cremona did not judge differently.

At the end I had risked a judgment on the non-Euclidean stereometry, which now I do not think correct. But the article, freed from this slip, will appear in the *Giornale* of Naples, in its original form, save certain additions that I can hazard now, because substantially concordant with some of the ideas of Riemann.''

The work referred to is the ''Saggio d' interpretazione della geometria non-euclidea'' (Giorn. di matem. 6, 1868, pp. 284-312), the first of the articles which, to use the expression of Cremona, ''gave to Beltrami as if at a cast that reputation which always widening became universal admiration.''

Its genial idea, for which Beltrami had been prepared by his researches on surfaces of constant curvature, an irresistable idea to silence opposers of the non-Euclidean geometry, was to present the example of a surface, regular as the plane and the sphere, in which the lines corresponding to the straights of the plane and to the great circles of the sphere, that is the geodetics, comported themselves as the straights of the non-Euclidean plane. This was a surface of constant negative curvature, a pseudosphere.

But very soon after the publication of this memoir, Helmholtz and Klein expressed grave doubts on certain points of Beltrami's reasoning and mathematics, saying that he had not made certain the existence of surfaces of the type to which he has recourse to represent, without changing its nature, the new system of geometry. It seems that Beltrami tried to dissipate these doubts. At least it seems that to such attempts we owe a memoir where is studied with scrupulous care the surface generated by the rotation of the tractrix about its asymptate with the aim of deducing the elements by a construction simple and exact of the surface itself.

The end aimed at not having been reached, Genocchi repeated the objections in a way still more particularized and energetic, maintaining that it had not been demonstrated that the partial differential equation, characteristic of the surface of constant negative curvature, admits at least one integral satisfying all the conditions imposed on the pseudo-sphere to serve the representation of Beltrami.

There is no trace of response from Beltrami, not even in his correspondence with Genocchi. Perhaps he perceived that unfortunately those objections were well founded.

But the final establishment of that fact, the complete demonstration of the non-existence of regular surfaces of constant negative curvature in all their extent, was not accomplished until after the death of Beltrami, by Hilbert in 1901.

The representation imagined by Beltrami is therefore sufficiently limited, and the beautiful edefice, if it does not fall to the ground, shows itself in solidity and extension less than what was believed. Fortunately the non-Euclidean geometry is now completely vindicated in many simpler ways. Moreover Beltrami's pseudo-sphere had always done harm to the many people who took up the false idea that geodesic geometry on the pseudo-sphere was Bolyai's non-Euclidean geometry, instead of only being an interesting *representation* of it in Euclidean space; just as Bolyai's geometry of goedesics on a limit surface in Lobachevski's space is a representation of Euclidean geometry in Bolyai space.

From this epoch (1868) the interest of Beltrami in non-Euclidean geometry never flagged.

He made an application of the expression given by Lobachevski for the angle of parallelism. In his correspondence with Genocchi he points out the flaw in the argumentation proposed by Carton for demonstrating the postulate of Euclid, presented by Bertrand to the Institute of France December 20, 1869, he discusses the enquiries undertaken to determine who was Schweikart, now so well known as an independent creator of the non-Euclidean geometry, and he gives an appreciation of Hoüel's article in "Sur l'impossibilité de démontrer par une construction plane le postulatum d' Euclide." A public proof of this interest is also the charming communication made to the Accademia dei Lincei March 17, 1889, to present in the proper light the work, whose value and significance were then unknown, of Saccheri, "Euclides ab omni naevo vindicatus."

Beltrami never ceased to meditate on the non-Euclidean geometry even when concentrating all his powers to the study of natural phenomena. A proof of this is his discovery that the general equation of elasticity is bound to the Euclidean postulate. Moreover one of his gifted disciples observes "how he shows, in a certain passage, that he had turned his attention to the way in which physics would be able to profit from hypotheses of a diverse geometric nature of space, a difficult conception, more explicitly advanced by Clifford; nor was he ever able to lose from view those curved spaces, with which he had commenced so triumphantly."

Settled finally at Rome, member of the most celebrated scientific societies of the world, successor to Brioschi as President of the Accademia dei Lincei, Senator of the Realm of Italy, many times chosen by public vote to sit in the Council Superior of Public Instruction, acclaimed master by the entire body of scientists, happy with a devoted wife, yet from 1896 he was undermined by a mysterious malady, and died February 18th, 1900.

The University of Texas.

ON THE PRIMITIVE GROUPS OF CLASS FOUR.

By DR. G. A. MILLER.

The class of a substitution group is the smallest number of elements in any one of its substitutions besides the identity. This definition seems to be due to Camille Jordan,* who made extensive investigations in regard to the primitive groups of small classes. It is well known that the alternating and the symmetric groups are of classes two and three respectively, and that these are the only primitive groups of these two classes. Moreover, Jordan proved that these are the only two classes for which the number of primitive groups is infinite,† and he investigated the problem of determining all the primitive groups whose class does not exceed 13, publishing only a brief outline of his work.‡

In his work on the *Theory of Substitutions*, pages 133 to 138, Netto gives an outline of a proof that there is no primitive group of class four and of degree greater than 8. As this theorem is of great importance in the theory of primitive groups, it appeared desirable to give a more complete proof, based upon some recent theorems. This proof will thus serve as another illustration of the application of these theorems, and it is hoped that it will tend to simplify one step towards the difficult subject of class of primitive substitution groups.

Let G be any primitive group of degree $n>8$ and of class four. The subgroup (H) generated by its substitutions of the form $ab.cd=s_1$, which may be supposed to be contained in G, must be invariant, since it includes all the con-

*Jordan, Liouville, vol. 16, 1871, page 383.
†Comptes Rendus, vol. 73, 1871, p. 853.
‡Loc. cit. vol. 75, p. 1757.

jugates of s_1. Hence it must be of degree n and transitive. It must be non-abelian since abelian transitive groups are regular. We shall first prove that H must include the regular four-group including s_1. This will prove that G is at least doubly transitive and that each one of its substitutions of degree four and order two is contained in a sub-group of order and degree four.

The sub-group H contains at least one substitution s_2 which is similar to s_1 and non-commutative with it. If s_1 and s_2 would have only one element in common their commutator $s_1^{-1}s_2^{-1}s_1s_2$ would be of degree three and G could not be of class four.* If they had three common elements they would generate a group of degree 5, which would be transitive since the intransitive groups of this degree are either of class 2 or of class 3. Every transitive group of degree p, p being any prime number, must include a cyclic substitution of degree p. It has been proved that such a substitution cannot occur in any primitive group whose class exceeds 3 except when the degree is one of the three numbers $p, p+1, p+2$.† Hence s_1 and s_2 must contain just two common elements; that is *any two non-commutative substitntions of G which are similar to s_1 have just two common elememts.*

The two substitutions s_1 and s_2 must therefore generate a dihedral rotation group of degree six whose order is either 6 or 8.‡ In the latter case s_1 is evidently included in a regular four-group. In the former case, the group generated by s_1 and s_2 is the intransitive group obtained by establishing a simple isomorphism between two symmetric groups of degree three. We proceed to prove that in this case s_1 must also be included in a regular four-group contained in H.

Only three of the substitutions of H which are similar to s_1 have been determined, viz: $s_1, s_2, s_1^{-1}s_2s_1$. If all the other similar substitutions were commutative with s_1 they would also have to be commutative with each of the other two given conjugates, since these three substitutions are transformed transitively by a sub-group which transforms all the rest among themselves. Hence H includes another substitution (s_3) which is not commutative with s_1, and therefore it has just two elements in common with s_1.

If the group generated by s_1, s_2, s_3 were transitive it would include a regular four-group containing s_1 since the degree of this group would be either six or seven and it may be assumed that it would not contain any substitution of order 5. As it would also be positive and of class four there are only two groups which require consideration, viz: $(+abcdef)_{24}$ and $(abcdefg)_{168}$.§ It remains to consider the cases when s_1, s_2, s_3 would generate one of the following intransitive groups: (1) A simple isomorphism between two symmetric groups of degree four, 2) A (4, 4) correspondence between these groups, and 3) A (1, 4) correspondence between the symmetric groups of degrees three and four respectively.

In the last two cases G would be at least doubly transitive and hence it

*Bochert, Mathematische Annalen, vol. 40, 1892, p. 159.

†Bulletin of the American Mathematical Society, vol. 4, 1898, p. 141.

‡Loc. cit., vol. 7, 1901, p. 424.

§American Jonrnal of Mathematics, vol. 21, p. 287.

would contain a substitution similar to s_1 which would permute its systems; that is, G would contain the regular four-group including s_1. In the first case G would contain an additional substitution (s_4) similar to s_1 and not commutative with s_1. If the group generated by s_1, s_2, s_3, s_4 were not a simple isomorphism between two symmetric groups of degree five it would clearly include a regular four-group containing a conjugate of s_1 and hence also such a group containing s_1. As this remark applies to all the following cases and as the substitutions which are similar to s_1 generate a transitive group, it follows that *in every primitive group of class four and degree* greater than 8 each substitution of type *ab.cd* is contained in the regular four-group. It may be observed that this applies also to the primitive groups of degrees 7 and 8 but not to those of degrees 5 and 6.

In what follows it may therefore be assumed that G is at least doubly transitive and that each of its substitutions similar to s_1 is contained in a regular four-group. It has already been observed that s_1, s_2 generate either the positive octic group of degree six or the intransitive group of degree and order six. In either case it may be assumed that s_1, s_2, s_3 generate $(+abcdef)_{24}$ since this is the only positive transitive group of degree six and class four which is generated by substitutions similar to s_1 and in which each of these substitutions is in a regular four-group. As G must contain some additional substitution (s_5) similar to s_1 which is not commutative with some one of the three conjugate substitutions of order 2 in $(+abcdef)_{24}$ and as the order of the group generated by s_1, s_2, s_3, s_5 must exceed 48, H must include a transitive group of degree seven or the alternating group of degree 6, the latter being the only positive group of degree six which includes $(+abcdef)_{24}$.

The only transitive group of degree seven and of class four is the well known simple group $(abcdefg)_{168}$. It follows from the theorem quoted above that this could not occur in a primitive group whose degree exceeds 9 unless this primitive group were either alternating or symmetric. It could not occur in a primitive group of degree 9 since all its 21 substitutions similar to s_1 are conjugate and hence each of these substitutions would be transformed into itself by $5.8=40$ substitutions of the primitive group of degree 9. This is clearly impossible since its order would not be divisible by 5. Hence there is no primitive group of class four and of degree greater than 8.

From what precedes and from the enumeration of the groups of degree 8* it follows that there are just six primitive groups of class four—two of each of the degrees 5 and 6 and one of each of the degrees 7 and 8. Their orders are 10, 20, 60, 120, 168, and 1344 respectively. The first two are the semi-metacyclic and the metacyclic groups† of degree 5. The third and fourth are, respectively, simply isomorphic with the alternating and the symmetric groups of degree six. They are the only instances of transitive groups of degree n and of orders $\frac{1}{2}(n-1)!$ and $(n-1)!$ and have been studied very fully in connection with the theory of equations of degree six. The first of these is known as the

*Cf. American Journal ol Mathematics, vol. 21, p. 287.

†Oeuvres de Lagrange, vol, 3, p. 339.

icosahedron rotation group and it is the smallest simple group of composite order. The fifth is very well known in the theory of elliptic modular functions and is the second smallest simple group of composite order. Kirkman remarks:* "Betti, Kronecker, Hermite, and myself have spent much time on this group." The last of these six primitive groups is the holomorph of the group of order 8 which includes no operator of order four.†

*Kirkman, Proceedings of the Manchester Literary and Philosophical Society, vol. 3, p. 65.
†American Journal of Mathematics, loc. cit.

Leland Stanford University.

FACTORS OF A CERTAIN DETERMINANT OF OREER SIX.

By DR. L. E. DICKSON.

The following is an example of the so-called Group-Determinant:*

$$D\equiv\begin{vmatrix} \mathrm{I} & \alpha & \beta & \gamma & \delta & \varepsilon \\ \beta & \mathrm{I} & \alpha & \delta & \varepsilon & \gamma \\ \alpha & \beta & \mathrm{I} & \varepsilon & \gamma & \delta \\ \gamma & \delta & \varepsilon & \mathrm{I} & \alpha & \beta \\ \delta & \varepsilon & \gamma & \beta & \mathrm{I} & \alpha \\ \varepsilon & \gamma & \delta & \alpha & \beta & \mathrm{I} \end{vmatrix}$$

Note that the elements of the first three rows form two cyclic determinants of order three, and that the elements of the last three rows form the same two cyclic determinants. It follows readily that D has the factors $(\mathrm{I}+\alpha+\beta)\pm(\gamma+\delta+\varepsilon)$.

Upon adding to the first column all the remaining columns, we obtain an equal determinant having $\mathrm{I}+\alpha+\beta+\gamma+\delta+\varepsilon$ throughout the first column. Let D_1 be the determinant obtained by removing this factor, so that the elements in the first column of D_1 are all unity. Subtracting the first row from the remaining rows, we find that

$$D_1=\begin{vmatrix} \mathrm{I}-\alpha & \alpha-\beta & \delta-\gamma & \varepsilon-\delta & \gamma-\varepsilon \\ \beta-\alpha & \mathrm{I}-\beta & \varepsilon-\gamma & \gamma-\delta & \delta-\varepsilon \\ \delta-\alpha & \varepsilon-\beta & \mathrm{I}-\gamma & \alpha-\delta & \beta-\varepsilon \\ \varepsilon-\alpha & \gamma-\beta & \beta-\gamma & \mathrm{I}-\delta & \alpha-\varepsilon \\ \gamma-\alpha & \delta-\beta & \alpha-\gamma & \beta-\delta & \mathrm{I}-\varepsilon \end{vmatrix}$$

From the first row, subtract the third, fourth, and fifth rows, and to the first row add the second row. In the resulting determinant, the elements of the first row are all divisible by $\mathrm{I}+\alpha+\beta-\gamma-\delta-\varepsilon$. Hence

*Its matrix forms the body of a left-hand multiplication-table for the symmetric group on three letters, where

I=identity, $\alpha=(123)$, $\beta=(132)$, $\gamma=(12)$, $\delta=(13)$, $\varepsilon=(23)$.

Compare Weber, *Algebra*, 2nd Edition, Vol, II, page 124.

$$D_1=(\mathrm{I}+\alpha+\beta-\gamma-\delta-\varepsilon)D_2,$$

where the elements in the first row of D_2 are 1, 1, -1, -1, -1; the remaining rows of D_2 being identical with the corresponding rows of D_1. Next subtract the first column of D_2 from the second column, and add the first column to the third, fourth, and fifth columns.

$$\therefore D_2=\begin{vmatrix} \mathrm{I}+\alpha-2\beta & -\alpha+\beta-\gamma+\varepsilon & -\alpha+\beta+\gamma-\delta & -\alpha+\beta+\delta-\varepsilon \\ \alpha-\beta-\delta+\varepsilon & \mathrm{I}-\alpha-\gamma+\delta & 0 & -\alpha+\beta+\delta-\varepsilon \\ \alpha-\beta+\gamma-\varepsilon & -\alpha+\beta-\gamma+\varepsilon & \mathrm{I}-\alpha-\delta+\varepsilon & 0 \\ \alpha-\beta-\gamma+\delta & 0 & -\alpha+\beta+\gamma-\delta & \mathrm{I}-\alpha+\gamma-\varepsilon \end{vmatrix}$$

Upon subtracting the second row from the first, the new first row is

$$\mathrm{I}-\beta+\delta-\varepsilon \quad -\mathrm{I}+\beta-\delta+\varepsilon \quad -\alpha+\beta+\gamma-\delta \quad 0$$

Add the fourth column to the first and the new first column to the second.

$$\therefore D_2=\begin{vmatrix} \mathrm{I}-\beta+\delta-\varepsilon & 0 & -\alpha+\beta+\gamma-\delta & 0 \\ 0 & \mathrm{I}-\alpha-\gamma+\delta & 0 & -\alpha+\beta+\delta-\varepsilon \\ \alpha-\beta+\delta-\varepsilon & 0 & \mathrm{I}-\alpha-\delta+\varepsilon & 0 \\ \mathrm{I}-\beta+\delta-\epsilon & \mathrm{I}-\beta+\delta-\epsilon & -\alpha+\beta+\gamma-\delta & \mathrm{I}-\alpha+\gamma-\varepsilon \end{vmatrix}$$

Subtract the first row from the third and fourth rows. In the resulting determinant, subtract the second column from the fourth column. Hence

$$D_2=\begin{vmatrix} \mathrm{I}-\beta+\delta-\epsilon & 0 & -\alpha+\beta+\gamma-\delta & 0 \\ 0 & \mathrm{I}-\alpha-\gamma+\delta & 0 & -\mathrm{I}+\beta+\gamma-\varepsilon \\ -\mathrm{I}+\alpha+\gamma-\delta & 0 & \mathrm{I}-\beta-\gamma+\epsilon & 0 \\ 0 & \mathrm{I}-\beta+\delta-\epsilon & 0 & -\alpha+\beta+\gamma-\delta \end{vmatrix}$$

By inspection, it is seen that this determinant is the square of

$$\triangle\equiv\begin{vmatrix} \mathrm{I}-\beta+\delta-\epsilon & -\alpha+\beta+\gamma-\delta \\ -\mathrm{I}+\alpha+\gamma-\delta & \mathrm{I}-\beta-\gamma+\varepsilon \end{vmatrix}$$

$$=\mathrm{I}^2+\alpha^2+\beta^2-\mathrm{I}\alpha-\mathrm{I}\beta-\alpha\beta-\gamma^2-\delta^2-\epsilon^2+\gamma\delta+\gamma\varepsilon+\delta\epsilon.$$

To show that $\triangle$ is algebraically irreducible, we note that

$$\triangle=(\mathrm{I}+\omega\alpha+\omega^2\beta)(\mathrm{I}+\omega^2\alpha+\omega\beta)-(\gamma+\omega\delta+\omega^2\epsilon)(\gamma+\omega^2\delta+\omega\varepsilon),$$

where ω is an imaginary cube root of unity. But the expressions

$$x=\mathrm{I}+\omega\alpha+\omega^2\beta,\ y=\mathrm{I}+\omega^2\alpha+\omega\beta,\ z=\gamma+\omega\delta+\omega^2\epsilon,\ w=\gamma+\omega^2\delta+\omega\varepsilon$$

are independent functions of I, α, β, γ, δ, ϵ. Hence $\triangle\equiv xy-zw$ is irreducible.

Hence the decomposition of the given determinant of order six into algebraically irreducible factors is as follows: *

$$D=(1+\alpha+\beta+\gamma+\delta+\varepsilon)(1+\alpha+\beta-\gamma-\delta-\varepsilon)\triangle^2.$$

In particular, D is expressible as the difference of two squares,

$$D=[(1+\alpha+\beta)\triangle]^2-[(\gamma+\delta+\epsilon)\triangle]^2.$$

The fact that $\triangle$ is a factor of D may be shown directly. Multiply the second row of D by ω and the third row by ω^2 and add the products to the first row. Multiply the fifth row by ω^2 and the sixth row by ω and add the products to the fourth row. The new first and fourth rows are

$$\begin{array}{cccccc} y & \omega y & \omega^2 y & z & \omega^2 z & \omega z, \\ w & \omega w & \omega^2 w & x & \omega^2 x & \omega x. \end{array}$$

Then develop the determinant by Laplace's method. Each term is the product of a determinant of order two formed from the above two rows by the complementary determinant of order four formed from the remaining rows. Each of these determinants of order two has the factor $xy-zw\equiv\triangle$. Hence D has the factor $\triangle$.

The University of Chicago, January, 1902.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

153. Proposed by J. C. CORBIN, Pine Bluff, Ark.

Find some two-figure numbers, such that if they be squared, then the figures interchanged and the resulting numbers squared, the resulting products will consist of the same digits in reversed order.

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The difference of the two numbers is a multiple of 9.

$\therefore$ $10x+y=10y+x+9n$.

$\therefore$ $x=y+n$.

Let $n=0$. The numbers are 11, 11; 22, 22.

*Since writing this paper, I find that the result was obtained in 1886 by Dedekind by the theory of matrices (Berliner Sitzungsberichte, 1897, page 1007).

Let $n=1$. The numbers are 21, 12.
Let $n=2$. The numbers are 31, 13.
Let $n=1.11$, $y=0$. The numbers are 10, .01.
Let $n=2.22$, $y=0$. The numbers are 20, .02.
Let $n=3.33$, $y=0$. The numbers are 30, .03.

Also solved by *J. H. DRUMMOND* and *J. K. ELLWOOD*.

154. Proposed by J. SCHEFFER, A. M., Hagerstown. Md.

Suppose there is a meadow of 8 acres in which the grass grows uniformly, and that 21 oxen could eat up the whole pasture in 6 weeks, or 18 oxen in 9 weeks; what number of oxen diminished by the removal of 9, at the end of 14 weeks, could eat it up in 18 weeks?

I. Solution by G. B. M. ZERR. A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

In the first case, in one week one ox will eat ($\frac{1}{6}$ of $\frac{8}{21}$ of original grass$+$ $\frac{8}{21}$ of what grows) on one acre$=\frac{4}{63}$ of original grass on one acre$+\frac{8}{21}$ of what grows on one acre.

In the second case, in one week one ox will eat ($\frac{1}{9}$ of $\frac{8}{18}$ of original grass $+\frac{8}{18}$ of what grows) on one acre$=\frac{4}{81}$ of original grass on one acre$+\frac{4}{9}$ of what grows on one acre.

In each case one ox eats the same quantity in one week.

$\therefore$ $\frac{4}{9}-\frac{8}{21}=\frac{4}{63}$ of the growth of one acre in one week is $=\frac{4}{63}-\frac{4}{81}=\frac{8}{576}$ of an acre. $\frac{8}{576}\div\frac{4}{63}=\frac{2}{9}$ of an acre, what grows on an acre in one week. $\frac{4}{63}+\frac{8}{21}$ of $\frac{2}{9}=\frac{4}{27}$, the part of the original quantity which one ox eats in one week.

$8\div(6\times\frac{4}{27})=9$ oxen to eat original grass. $21-9=12$ oxen necessary to eat growing grass.

Now change "at the end of 14 weeks" to "at the end of 4 weeks." Otherwise we will have same hungry oxen. 8 acres$=\frac{2}{9}\times18\times8=40$ acres of original grass$=$amount on 8 acres$+$what grows on 8 acres in 18 weeks.

Then $4\times\frac{4}{27}$ oxen$+14\times\frac{4}{27}$ (oxen-9)$=40$.

$\therefore$ 72 oxen$=1080+504=1584$. $\therefore$ oxen$=22$, the number required.

Also solved by *J. R. HITT*.

II. Solution by the PROPOSER.

Let a be the number of pounds of grass on 1 acre, and an that which grows on 1 acre in 1 week. Then 21 oxen eat $(8a+8an\times6)$ pounds in 6 weeks, and 18 oxen eat $(8a+8an\times9)$ pounds in 9 weeks.

$\therefore$ From the first statement, 1 ox eats $\dfrac{4(1+6an)}{63}$ pounds in 1 week.

From the second statement, 1 ox eats $\dfrac{4(1+9an)}{81}$ pounds.

$\therefore \dfrac{4(a+9an)}{81}=\dfrac{4(a+6an)}{63}$, whence $n=\frac{2}{9}$.

Let x=the required number of oxen. Then x oxen will eat $\dfrac{56ax}{27}$ pounds in 14 weeks, and $(x-9)$ oxen will eat $\frac{16}{27}a(x-9)$ pounds in 4 weeks.

$$\therefore \frac{56ax}{27}+\frac{16a}{27}(x-9)=8(a+18an)=40a, \text{ whence } x=22, \text{ number required.}$$

Also solved by *J. R. HITT*.
Solutions of problem 152 were received from *J. K. ELLWOOD* and *J. R. HITT*.

ALGEBRA.

NOTE ON SOLUTION I. OF PROBLEM 131.

In connection with this solution it is well to note that Dr. Zerr's results hold only for the case where a, b, c, and d are all positive, and a^2d+b^2c-cd is positive, restrictions which he neglects to mention. For instance, the equation $2x-1+\sqrt{(5x^2+1)}=0$ possesses two roots, 0 and -4, yet $c>a^2$ or $5>2^2$.

The criteria of the second solution cover all real values of a, b, c, and d, positive or negative, and a^2d+b^2c-cd positive.

H. S. VANDIVER, Bala, Pa.

133. Proposed by HARRY S. VANDIVER, Bala, Pa.

A theory of Fermat. The sum of two integral fourth powers cannot be an integral square. [Cf. *Chrystal's Algebra*, Vol. II, page 535.]

I. Solution by L. C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma Cal., and CHAS. C. CROSS, Whaleyville, Va.

Assume m and n to be the numbers, then by the condition of the problem, we have

$$(m^2)^2+(n^2)^2=c^2, \text{ suppose......(1)}.$$

In (1) either m^2 or n^2 must be even, and the other odd, because the sum of two odd squares can not be a square.

Assume m^2 even and n^2 odd.

Let $m^2=2pq$ and $n^2=p^2-q^2$; or $n^2+q^2=p^2$......(2).

In (2) n and p are odd and q even. Now let $q=2\alpha\beta$ and $p=\alpha^2+\beta^2$. Then we find $m^2=4\alpha\beta(\alpha^2+\beta^2)$......(3).

Since α and β are prime to each other, in order that $\alpha\beta$ may be a square, each must be a square.

Let $\alpha=m_1^2$ and $\beta=n_1^2$. Substituting in (3), we get

$$m^2=4m_1^2n_1^2(m_1^4+n_1^4)\text{......(4)}.$$

In order that the right member of (4) may be a square, we must have $m_1^4+n_1^4=c_1^2$, say; which is of the same form as (1). But c_1^2 is less than c^2.

Proceeding in exactly the same way, we can reduce c^2 indefinitely by some integer. By our hypothesis c^2 can not be zero, nor less.

Hence, results *Fermat's Theorem.*

Also solved by *G. B. M. ZERR*.

II. Remarks by the PROPOSER.

Euler's proof of this theorem which appears in his treatise on Algebra, I believe to be the simplest demonstration that can possibly be obtained. For the benefit of those who are unacquainted with it, I reproduce the substance of the argument below.

EULER'S PROOF. Suppose that three integers a, b, and c are found such that $a^4+b^4=c^2$. If a and b are not prime to each other let $a=a'y$, and $b=a'x$, then $(a'y)^4+(a'x)^4=c^2$. Whence, by division, $x^4+y^4=\square=z^2$, suppose.

Hence it is sufficient to consider the case

$$x^4+y^4=z^2\ldots\ldots(1),$$

where x, y and z are prime to each other. If this relation be satisfied, then it follows that there are integers p and q such that

$$\left.\begin{array}{l} x^2=2pq \\ y^2=p^2-q^2 \end{array}\right\} \text{ or } \left\{\begin{array}{l} x^2=p^2-q^2 \\ y^2=2pq \end{array}\right.$$

Since x and y are symmetrical in (1) it is sufficient to put

$$x^2=2pq\ldots\ldots(2),\ y^2=p^2-q^2\ldots\ldots(3).$$

Since x is prime to y and x is evidently even, it follows that y is odd, and y^2 must then be of the form $4n+1$. In (3) we have the following hypotheses for p^2 and q^2 :

$$\begin{array}{l} p^2=4m \text{ or } 4m+1 \\ q^2=4k \text{ or } 4k+1 \end{array}$$

If $p^2=4m$ and $q^2=4k$ then $y^2=4(m-k)$, which is impossible, since y is odd. Put $p^2=4m$ and $q^2=4k+1$; then $y^2=4(m-k)-1$, which is also absurd.

$\therefore$ $p^2=4m+1$ and $q^2=4k$, whence p is odd and q is even.

Therefore integers r and s can be found so that $p=r^2+s^2$, $q=2rs$, where r and s are prime to each other.

Substituting in (2) we have

$$x^2=4rs(r^2+s^2) \text{ or } x_1^2=rs(r^2+s^2).$$

Since r, s and r^2+s^2 are all prime to each other we will have $r=r_1^2$, $s=s_1^2$. Then also, $r^2+s^2=r_1^4+s_1^4=\square=u_1^2$, say. Now in $r_1^4+s_1^4=u_1^2$, $r_1<r<x$.

That is, supposing (1) to hold, we have shown that it is possible to find a relation $r_1^4+s_1^4=u_1^2$ so that $r_1<x$.

In the same manner it may be shown that we can obtain $r_2^4+s_2^4=u_2^2$, where $r_2<r_1$, and evidently, $r_n^4+s_n^4=u_n^2$ where $r_n<r_{n-1}<r_{n-2}\ldots\ldots<r_1<x$.

So that by taking n sufficiently large it will be found

$$1+s_n{}^4=u_n{}^2\ldots\ldots(4),$$

whence $u_n{}^2-s_n{}^4=1$, and $u_n=1$ and $s_n=0$ are the only solutions. Now s_n cannot be 0 for if such were the case we would have

$$s_n=s_{n-1}=s_{n-2}\ldots\ldots=s_1=x=0,$$

which is inconsistent with the definition of x. Hence the impossibility of (4) and therefore of (1) is completely demonstrated.

134. **Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.**

Solve neatly and briefly the equations

$$x^3+x^2y+y^3=53\ldots.(1),\ y^3+y^3z+z^3=13\ldots.(2),\text{ and } z^3+z^2x+x^3=31\ldots.(3).$$

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Probably the only brief solution is by inspection, as follows:

$$x^3+x^2y+y^3=53=27+18+8=3^3+2.3^2+2^3.$$

$$y^3+y^2z+z^3=13=8+4+1=2^3+1.2^2+1^3.$$

$$z^3+z^2x+x^3=31=1+3+27=1^3+3.1^2+3^3.$$

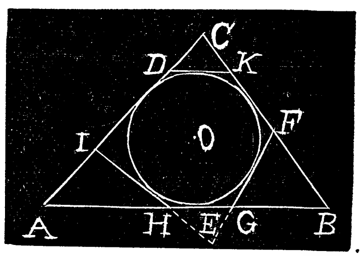
$\therefore\ x=3,\ y=2,\ z=1.$

135. **Proposed by CHARLES C. CROSS, Whaleyville, Va.**

Tangents parallel to the three sides are drawn to the in-circle. If p, q, r, be the lengths of the parts of the tangents within the triangle, prove that

$$\frac{p}{a}+\frac{q}{b}+\frac{r}{c}=1.$$

I. **Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.**



Let ABC be any triangle with the in-circle O. Put $AB=c$, $AC=b$, $BC=a$.

Draw the respective tangents, $DK=r$, parallel to AB; $FG=q$, parallel to AC; and $HI=p$, parallel to BC.

Let $AH=x$, and $BG=y$.

The construction of lines and similarity of triangles give the following:

$HG=DE=r$; $y:c=q:b$, or $y=cq/b$;

and $x:c=p:a$, or $x=cp/a$. But $x+y+r=c$.

$$\therefore\ \frac{cp}{a}+\frac{cq}{b}+r=c;\text{ or }\frac{p}{a}+\frac{q}{b}+\frac{r}{c}=1.$$

Solved in a similar manner by *LON C. WALKER* and *J. SCHEFFER*.

II. Solution by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

Designate the perpendiculars from A, B, C by h', h'', h'''; then from similar triangles,

$$p':a::h'-2r:h', \text{ or } \frac{p'}{a}=1-2\frac{r}{h'}.$$

Similarly, $\frac{p''}{b}=1-2\frac{r}{h''}$ and $\frac{p'''}{c}=1-2\frac{r}{h'''}$; whence

$$\frac{p'}{a}+\frac{p''}{b}+\frac{p'''}{c}=3-2\left(\frac{r}{h'}+\frac{r}{h''}+\frac{r}{h'''}\right)=1\ldots\ldots(1),$$

$$\left(\text{since, } \frac{1}{h'}+\frac{1}{h''}+\frac{1}{h'''}=\frac{1}{r}\right).$$

Similarly, if P', P'', P''', denote the length of the tangent to the *escribed* circles

$$\frac{P'}{a}+\frac{P''}{b}+\frac{P''}{c}=3+2\left(\frac{r'}{h'}+\frac{r''}{h''}+\frac{r'''}{h'''}\right)\ldots\ldots(2).$$

Again, for a *tetrahedron*, let p', p'', p''', p'''', designate the planes drawn tangent to the inscribed sphere and parallel to the faces a, b, c, d; and P', P'', P''', P'''' the plane similarly drawn tangent to the escribed sphere. Then

$$\sqrt{\frac{p'}{a}}+\sqrt{\frac{p''}{b}}+\sqrt{\frac{p'''}{c}}+\sqrt{\frac{p''''}{d}}=4-2\left(\frac{r}{h'}+\frac{r}{h''}+\frac{r}{h'''}+\frac{r}{h''''}\right)=2\ldots\ldots(3).$$

$$\sqrt{\frac{P'}{a}}+\sqrt{\frac{P''}{b}}+\sqrt{\frac{P'''}{c}}+\sqrt{\frac{P''''}{d}}=4+2\left(\frac{r'}{h'}+\frac{r''}{h''}+\frac{r'''}{h'''}+\frac{r''''}{h''''}\right)\ldots\ldots(4).$$

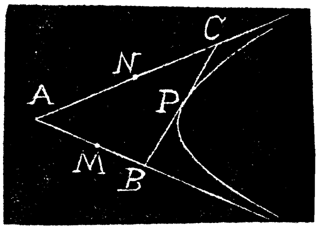
Also solved by *G. B. M. ZERR* and *H. C. WHITAKER*.

GEOMETRY.

164. Proposed by J. M. HARCOURT, M. D., 305 Clinton Street, Brooklyn, N. Y.

Given two tangents to a parabola, find the locus of the center of the nine-point circle of the triangle by the two given tangents and any third tangent.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.



Let the equation of the parabola be $y^2=4mx$; then will the equation of three tangents be of the form $y=ax+\frac{m}{a}\ldots\ldots(1)$, $y=a'x+\frac{m}{a'}\ldots\ldots(2)$, $y=a''x+\frac{m}{a''}\ldots\ldots(3)$; the equation (1) being that of AB, (2) of AC, and (3) of BC.

The equation of the altitude from A upon

BC will be

$$aa'(x+a''y)=m(1+a'a''+a''a)\ldots\ldots(4),$$

and that of C upon AB,

$$a'a''(x+ay)=m(1+a''a+aa'')\ldots\ldots(5).$$

Combining (4) and (5), we find the co-ordinates of the orthocenter of $\triangle ABC$ to be $x=-m$, $y=m\left(\frac{1}{aa'a''}+\frac{1}{a}+\frac{1}{a'}+\frac{1}{a''}\right)\ldots\ldots(6)$.

The symmetry with reference to a, a', a'' shows that the three altitudes all pass through a single point, and the value of x shows that the locus of the orthocenter of *any* variable tangential triangle whatever is the directrix of the parabola. Since the co-ordinates of A, B, and C are respectively,

$$\frac{m}{aa'},\ m\left(\frac{1}{a}+\frac{1}{a'}\right);\ \frac{m}{aa''},\ m\left(\frac{1}{a}+\frac{1}{a''}\right);\ \frac{m}{a'a''},\ m\left(\frac{1}{a'}+\frac{1}{a''}\right)\ldots\ldots(7),$$

we find for the middle points M, N, P of the sides AB, AC, BC, respectively, the co-ordinates

$$\frac{m}{2a}\left(\frac{1}{a'}+\frac{1}{a''}\right),\ \frac{m}{2}\left(\frac{2}{a}+\frac{1}{a'}+\frac{1}{a''}\right);\ \frac{m}{2a'}\left(\frac{1}{a}+\frac{1}{a''}\right),\ \frac{m}{2}\left(\frac{2}{a'}+\frac{1}{a}+\frac{1}{a''}\right);$$

$$\frac{m}{2a''}\left(\frac{1}{a}+\frac{1}{a'}\right),\ \frac{m}{2}\left(\frac{2}{a''}+\frac{1}{a}+\frac{1}{a'}\right)\ldots\ldots(8).$$

Consequently the equation of the perpendicular erected at M and N, respectively, is

$$y+\frac{1}{a}x=\frac{m}{2}\left[\frac{2}{a}+\left(1+\frac{1}{a^2}\right)\left(\frac{1}{a'}+\frac{1}{a''}\right)\right]$$

$$\text{and } y+\frac{1}{a'}x=\frac{m}{2}\left[\frac{2}{a'}+\left(1+\frac{1}{a'^2}\right)\left(\frac{1}{a}+\frac{1}{a''}\right)\right]\ \ldots\ldots(9).$$

Therefore, co-ordinates of their point of intersection, or the center of the circumcircle

$$x=\frac{m}{2}\left(1+\frac{1}{aa'}+\frac{1}{aa''}+\frac{1}{a'a''}\right),\ y=\frac{m}{2}\left(\frac{1}{a}+\frac{1}{a'}+\frac{1}{a''}-\frac{1}{aa'a''}\right)\ldots\ldots(10).$$

Since the center of the nine-point circle is the middle point of the straight line that connects the center of the circumcircle with the orthocenter, we find for the co-ordinates of the center of the nine-point circle

$$x=\frac{m}{4}\left(-1+\frac{1}{aa'}+\frac{1}{aa''}+\frac{1}{a'a''}\right),\ y=\frac{m}{4}\left[3\left(\frac{1}{a}+\frac{1}{a'}+\frac{1}{a''}\right)+\frac{1}{aa'a''}\right]..\ (11)$$

Considering AB and CD the given tangents, therefore a and a' given quantities, and a'' variable, we have in (11) only to eliminate a'' to obtain the equation of the required locus. Thus, putting for the sake of brevity $1/a+1/a'=a$, $1/aa'=b$, we find the equation of the locus to be

$$4ay-4(3-b)x=m[3(1+a^2)-b(4-b)].$$

Consequently, the locus is a straight line.

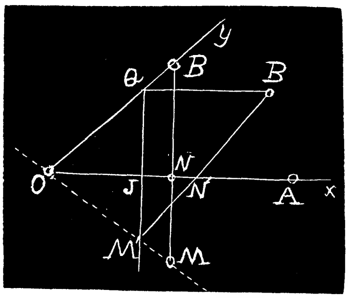
Also solved by *G. B. M. ZERR.*

165. **Proposed by W. H. ECHOLS. B.Sc.. S.E., Professor of Mathematics. University of Virginia. Charlottsville, Va.**

$OB=b$, $OA=a$ are the semi-conjugate diameters of an ellipse. Draw BM perpendicular to and equal to OA, cutting it in N. Show that as M slides on the fixed line OM and N on OA the point B traces the curve.

Solution by the PROPOSER.

Let $M'N'B'$ be an arbitrary position of the line. Draw $B'Q$ parallel to Ox, join $M'Q$. Then



$$\frac{M'J}{JQ}=\frac{M'N'}{N'B'}=\frac{MN}{NB}.$$

$\therefore$ $M'Q$ is perpendicular to Ox.
Hence, if $B'Q=x$, $OQ=y$, we have

$$x^2+(M'Q)^2=a^2, \text{ or } x^2+\frac{a^2}{b^2}y^2=a^2.$$

$\therefore$ $x^2/a^2+y^2/b^2=1$. Q. E. D.

166. **Proposed by S. F. NORRIS. Professor of Astronomy and Mathematics. Baltimore City College, Baltimore, Md.**

Two cities are 200 miles apart. To what height must a man ascend from one city in order that he may see the other, supposing the circumference of the earth to be 25,000 miles? [From Wentworth's *New Plane and Solid Geometry*, page 381, No. 619.] Required solution by Geometry.

Solution by DANIEL NORTHROP. Mandana, N. Y.

Let A be the position of one city and B the position of the second, distant on a straight line from A, $a=200$ miles. Let P be the position of the man above A when just able to see the city B, h his height above A. Draw the line PA and extend it through the center of the earth to the point D opposite A. Draw the line PB. Then we have $PB^2=PD\times PA$, or $PB^2=(h+2R)h$.

From the similar triangles PAB and PBD, we have $PB:DB=PA:AB$, or $PB:DB=h:a$. But $DB=\sqrt{(AD^2-AB^2)}=\sqrt{(4R^2-a^2)}$.

$\therefore PB:\sqrt{(4R^2-a^2)}=h:a$. Whence $PB=\dfrac{h\sqrt{(4R^2-a^2)}}{a}$.

$$\therefore \frac{h^2\,(4R^2-a^2)}{a^2}=h(h+2R).$$

$$\therefore h=\frac{2Ra^2}{4R^2-2a^2}=\frac{Ra^2}{2R^2-a^2}=5.0062 \text{ miles.}$$

Also solved by *G. B. M. ZERR*, *L. C. WALKER*, *H. C. WHITAKER*, and *J. SCHEFFER*.

CALCULUS.

127. Proposed by J. A. CALDERHEAD, B.Sc., Professor of Mathematics, Curry University, Pittsburg. Pa.

Find the moment of inertia of a parallelogram about an axis perpendicular to its plane and passing through the intersection of its diagonals.

Solution by G. B. M. ZERR, A.M.. Ph. D., The Temple College. Philadelphia, Pa.; J. SCHEFFER, A. M., Hagerstown, Md., and J. M. ARNOLD, Crompton, R. I.

Let the intersection of the diagonals be the origin; straight lines through the origin parallel to the sides, the axes; a, b the sides.

Then the required moment of inertia is

$$I=\rho\sin\beta\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\int_{-\frac{1}{2}b}^{\frac{1}{2}b}[(x+y\cos\beta)^2+y^2\sin^2\beta]dxdy$$

where β is the acute angle of the parallelogram.

$$\therefore I=\rho\sin\beta\int_{-\frac{1}{2}a}^{\frac{1}{2}a}(bx^2+\tfrac{1}{12}b^3)dx=\tfrac{1}{12}\rho ab\sin\beta(a^2+b^2)=\tfrac{1}{12}m(a^2+b^2).$$

Also solved by *W. J. GREENSTREET* and *L. C. WALKER*.

128. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

The differential equation of a curve is $\dfrac{d^3y}{dx^2}+y=0$. Find its equation, there being the additional conditions that for $x=0$, $y=1$, that the tangent at the point (0, 1) makes an angle of 45° with the axes, and finally that that point is a point of inflexion.

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy. Ohio University, Athens, O.; and L. C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Using D for the operator d/dx, the given equation is

$$(D^3+1)y=0\ldots\ldots(1), \text{ or } (D^2-D+1)(D+1)y=0\ldots\ldots(2).$$

Integrating $(D+1)y=e^{\frac{1}{2}x}(A\cos\frac{\sqrt{3}}{2}x+B\sin\frac{\sqrt{3}}{2}x)$......(3).

Differentiating this with respect to x,

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}=e^{\frac{1}{2}x}[(\tfrac{1}{2}\cos\frac{\sqrt{3}}{2}x-\frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}x)A$$

$$+(\tfrac{1}{2}\sin\frac{\sqrt{3}}{2}x+\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}x)B]......(4).$$

When $x=0$, $y=1$, $dy/dx=1$, and (3) gives $A=2$.
When $x=0$, $D=1$, $D^2=0$, and (4) gives $B=0$.
$\therefore$ (3) is $(D+1)y=2e^{\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x$......(5).

Integrating (5), and noticing that when $x=0$, and $y=1$, the constant of integration is nought,

$$y=e^{\frac{1}{2}x}(\tfrac{1}{3}\sqrt{3}\sin\frac{\sqrt{3}}{2}x+\cos\frac{\sqrt{3}}{2}x)......(6).$$

Also solved by *G. B. M. ZERR* and *J. SCHEFFER*.

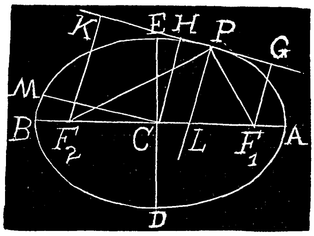
MECHANICS.

128. Proposed by M. E. GRABER, A. B., Heidelberg University, Tiffin, Ohio.

A particle is placed on the convex side of a smooth ellipse and is acted upon by two forces, F and F', towards the foci, and a force, F'', towards the center. Find the position of equilibrium.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; and the PROPOSER.

Let C be the center of the ellipse, semi-axes a, b; F_1, F_2, the foci; force $PF_1=F$, $PF_2=F'$, $PC=F''$; distance $PC=r$. Let fall the perpendiculars F_1G, CH, F_2K from the foci F_1, F_2, and the center C on the tangent at P.



Let $\angle PF_1G=\angle PF_2K=\theta$, $\angle PCH=\varphi$.

Then $F\sin\theta=F'\sin\theta+F''\sin\varphi$.

$\therefore \dfrac{\sin\varphi}{\sin\theta}=\dfrac{F-F'}{F''}=n$, suppose.

$F_2K=F_2P\cos\theta$, $F_1G=F_1P\cos\theta$, $CH=r\cos\varphi$.

$2CH=F_2K+F_1G=(F_2P+F_1P)\cos\theta=2a\cos\theta$.

$\therefore r\cos\varphi=a\cos\theta$.

$F_2K.F_1G=b^2=F_2P.F_1P\cos^2\theta=CM^2\cos^2\theta=(a^2+b^2-r^2)\cos^2\theta$.

$$\therefore \cos\theta=\frac{b}{\sqrt{(a^2+b^2-r^2)}},\quad \sin\theta=\frac{\sqrt{(a^2-r^2)}}{\sqrt{(a^2+b^2-r^2)}}.$$

$$\therefore \cos\varphi = \frac{ab}{r\sqrt{(a^2+b^2-r^2)}},\ \sin\varphi = \frac{\sqrt{[r^2(a^2+b^2-r^2)-a^2b^2]}}{r\sqrt{(a^2+b^2-r^2)}}.$$

$$\therefore\ n = \frac{\sqrt{[r^2(a^2+b^2-r^2)-a^2b^2]}}{r\sqrt{(a^2-r^2)}} = \frac{\sqrt{(r^2-b^2)}}{r}.$$

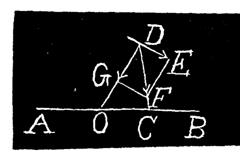
$$\therefore\ r^2-b^2=r^2n^2,\ \text{or}\ r=\frac{b}{\sqrt{(1-n^2)}}.$$

129. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

Two spheres whose masses are M_1 and M_2 are a units apart, and attract each other with a force $=M_1M_2/a_2$. Find work done in carrying a unit mass from the center point between them a distance r in a direction θ with line of centers.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let $B=m_1$, $A=m_2$, $m_1>m_2$; C, the centroid of m_1, m_2; $OC=c$, $OD=x$, $DC=y$, $\angle DOC=\theta$, $\angle OCD=\varphi$.



Then $y=\sqrt{(c^2+x^2-2cx\cos\theta)}$, $c=\dfrac{a(m_1-m_2)}{2(m_1+m_2)}$.

$c=y\cos\varphi+x\cos\theta$. $\therefore\ \cos\varphi=\dfrac{c-x\cos\theta}{y}$.

Force acting on $D=(m_1+m_2)/y^2$.

Resolving this force, the component along OD

$$=(m_1+m_2)\cos(\theta+\varphi)/y_2=\frac{m_1+m_2}{y^3}\{(c-x\text{sos}\theta)\cos\theta-\sin\theta\sqrt{[y^2-(c-x\cos\theta)^2]}\}$$

$$=\frac{(m_1+m_2)(c\cos\theta-x)}{(c^2+x^2-2cx\cos\theta)^{\frac{3}{2}}}.$$

$$\text{Work}=(m_1+m_2)\int_0^r \frac{(c\cos\theta-x)dx}{(c^2+x^2-2cx\cos\theta)^{\frac{3}{2}}}$$

$$=(m_1+m_2)\left(\frac{1}{\sqrt{(c^2+r^2-2cr\cos\theta)}}-\frac{1}{c}\right)$$

$$=2(m_1+m_2)^2\left(\frac{1}{\sqrt{4r^2(m_1+m_2)^2+a^2(m_1-m_2)^2-4ar(m_1^2-m_2^2)\cos\theta}}-\frac{1}{m_1-m_2}\right).$$

DIOPHANTINE ANALYSIS.

86. Proposed by B. F. FINKEL, A. M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Prove that the congruence $x^2=1457\equiv0(\text{mod}2389)$ is not possible.

A solution of this problem is given in Mathews' *Theory of Numbers*, page 42. ED.

87. Proposed by LON C. WALKER. A.M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Find three numbers in arithmetical progression the sum of whose cubes is a cube.

Solution by the PROPOSER.

Let tx, ty, tz represent the three required numbers. Then we have

$t^3x^3+t^3y^3+t^3z^3=w^3$; or, $x^3+y^3+z^3=w^3/t^3=v^3$......(1), assume.

Transposing (1), we have to satisfy the equation

$$v^3-x^3-y^3=z^3......(2).$$

Put $x=a(r-q)$, $y=aq$, $z=a[p-(rq^2/p^2)]$, and $v=ap$. Then we have, after dividing by a^3,

$$p^3-(r-q)^3-q^3=(p-\frac{rq^2}{p^2})^3......(3).$$

Whence, by involution and reduction,

$$r=\frac{3p^3q}{p^3+q^3}.$$

Now take $a=p^3+q^3$, and we have

$$x=q(2p^3-q^3),\ y=q(p^3+q^3),\ z=p(p^3-2q^3),\ v=p(p^3+q^3),$$

where p and q may have any values subject to the condition $p^3>2q^3$.

If we take $p=2$, $q=1$, after dividing all by 3, we have, $3^3+4^3+5^3=6^3$, the least three positive integral numbers that satisfy conditions of the problem.

[See *Mathematical Magazine*, page 154, from whence the above solution was copied.]

88. Proposed by L. C. WALKER. A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Find three square numbers in harmonical progression.

Solution by HARRY S. VANDIVER, Bala, Pa.

Solutions of this problem have been given on pages 82-83 of Vol. VII, of the MONTHLY, substantially as follows:

Let $1/a^2$, $1/b^2$, $1/c^2$, be the numbers. Then $a^2+c^2=2b^2$, and solutions are given by

$$a=m^2-n^2+2mn$$
$$c=n^2-m^2+2mn$$
$$b=m^2+n^2.$$

To make these solutions complete, however, it is necessary to show that the formulae for a, c, and b give all the possible solutions. This may be proved as follows:

If $a^2+c^2=2b^2$ then b must be of the form m^2+n^2, for every divisor of the sum of two squares is itself the sum of two squares. Now

$$2b^2=2(m^2+n^2)^2=(1+1)(m^2+n^2)^2$$

can be expressed as the sum of two squares in but one way, the following:

$$(m^2-n^2-2mn)^2+(m^2-n^2+2mn).$$

Hence $a=m^2-n^2+2mn$, and $c=n^2-m^2+2mn$ give all the solutions of the problem.

Also solved by *L. C. WALKER.*

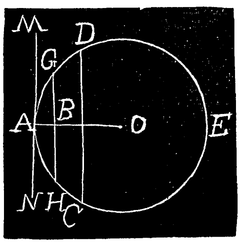
AVERAGE AND PROBABILITY.

109. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

A cylinder pierces a sphere in such a manner that the cylinder is tangent internally to the projection of the sphere in the plane xy. Find (1) the average surface, (2) the average volume of the sphere included within the cylinder.

Solution by LON C. WALKER, A.M., Professor of Mathematics, Petaluma High School. Petaluma, Cal.

Let $ACED$ be a section of the sphere through the center and the axis of the cylinder, O the center of the sphere, GH the axis of the cylinder, MN an element of the cylinder tangent to the sphere at A. Let S, V, be the required average surface and volume, respectively.



Put $OA=a$, $AB=r$. Now if we take AO for the axis of x, AM for the axis of z, and the perpendicular to OAM at A for the axis of y, the equations of the sphere and cylinder are respectively,

$$x^2+y^2+z^2=2ax, \text{ and } x^2+y^2=2rx.$$

$$\therefore\ S=\frac{2\int_0^a\int_0^{2r}\int_0^{\sqrt{(2rx-x^2)}}\frac{adrdxdy}{\sqrt{(2ax-x^2-y^2)}}}{\int_0^a dr}=\pi\int_0^a\int_0^{2r}drdx=\pi a^2,$$

the area of a great circle of the sphere.

If $y'=\sqrt{(2ax-x^2)}$, $z=\sqrt{(2ax-x^2-y^2)}$, then we have

$$V=\frac{4}{a}\int_0^a\int_0^{2r}\int_0^{y'}\int_0^{z'}drdxdydz=\frac{4}{a}\int_0^a\int_0^{2r}\int_0^{y'}(2ax-x^2-y^2)drdxdy$$

$$=\frac{2}{a}\int_0^a\int_0^{2r}\left[(2ax-x^2)\sin^{-1}\left(\frac{2r-x}{2a-x}\right)^{\frac{1}{2}}+x\sqrt{[2(a-r)(2r-x)]}\right]drdx$$

$$=\frac{8}{9a}\int_0^a\left[3a^3\tan^{-1}\left(\frac{r}{a-r}\right)^{\frac{1}{2}}-(3a^2+2ar-8r^2)\sqrt{(ar-r^2)}\right]dr=\tfrac{5}{6}a^3\pi$$

$=\frac{5}{8}$ of the volume of the sphere.

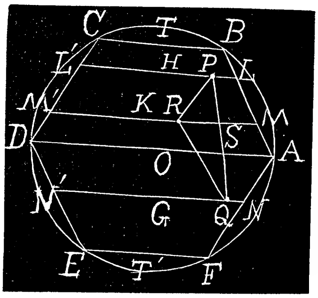
Also solved by *G. B. M. ZERR*, who gets $\frac{1}{2}\pi R^3$ as a result for the second part of the problem.
A partial solution was received from F. P. Matz.
Professor Walker should have received credit in the last issue for solution of problem 108.

110. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Find the average area of the triangle formed by joining three random points taken on the surface of a regular hexagon, two on one side of a diagonal and the third on the other side.

Solution by the PROPOSER.

Let $ABCDEF$ be the hexagon; P, R the random points above the diagonal AD; Q the random point below the diagonal. Through P, R, Q draw LL', MM', NN' parallel to AD, and TT' perpendicular to AD through O. It is only necessary to consider the relative positions in which the line MM' lies between LL' and NN'.



Let $CB=OA=a$ be the side of the hexagon, $OH=u$, $OG=v$, $OK=w$, $HP=x$, $GQ=y$, $KR=z$, $KS=t$, $HL=x'$, $GN=y'$, $KM=z'$.

Then $OT=\frac{1}{2}a\sqrt{3}=u'$, $x=(a\sqrt{3}-u)/\sqrt{3}$, $y'=(a\sqrt{3}-v)/\sqrt{3}$, $z'=(a\sqrt{3}-w)/\sqrt{3}$, $t=y-[(y-x)(v+w)]/(u+v)$.

Area $PQR=\frac{1}{2}(t-z)(u+v)=A$, when $t>z$.

Area $PQR=\frac{1}{2}(z-t)(u+v)=A_1$, when $t<z$.

The limits of u are 0 and $\frac{1}{2}a\sqrt{3}$; of v, 0 and $\frac{1}{2}a\sqrt{3}$; of w, 0 and u; of x, $-x'$ and x'; of y, $-y'$ and y'; of z, $-z'$ and t, and t and z'.

The whole number of ways the three points can be taken is $\frac{81\sqrt{3}}{64}a^6$. Doubling, since the halves are interchangeable, we get for average area of triangle:

$$\triangle=\frac{128}{81\sqrt{3}a^6}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^u\int_{-x'}^{x'}\int_{-y'}^{y'}\left[\int_{-z'}^{t}Adz+\int_t^{z'}A_1dz\right]dudvdwdxdy$$

$$=\frac{64}{81\sqrt{3}a^6}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^u\int_{-x'}^{x'}\int_{-y'}^{y'}\left[\tfrac{1}{3}(a\sqrt{3}-w)^2+\left(y-\frac{(y-x)(v+w)}{u+v}\right)^2\right]$$

$$\times(u+v)dudvdwdxdy$$

$$=(\tfrac{2}{3})^7\frac{1}{a^6}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^u\int_{-x'}^{x'}\Big[3(a\sqrt{3}-w)^2(a\sqrt{3}-v)(u+v)$$

$$+\frac{(a\sqrt{3}-v)^3(u-w)^2+9x^2(a\sqrt{3}-v)(v+w)^2}{u+v}\Big]dudvdwdx$$

$$=(\tfrac{2}{3})^8\frac{\sqrt{3}}{a^6}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^u\Big[3(a\sqrt{3}-u)(a\sqrt{3}-v)(a\sqrt{3}-w)^2(u+v)$$

$$+\frac{(a\sqrt{3}-u)(a\sqrt{3}-v)^3(u-w)^2+(a\sqrt{3}-u)^3(a\sqrt{3}-v)(v+w)^2}{u+v}\Big]dudvdw$$

$$=(\tfrac{2}{3})^8\frac{1}{\sqrt{3}a^6}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^{\frac{1}{2}(a\sqrt{3})}\Big[9a^3\sqrt{3}(a\sqrt{3}-u)(a\sqrt{3}-v)(u+v)$$

$$-3(a\sqrt{3}-u)^4(a\sqrt{3}-v)(u+v)+(a\sqrt{3}-u)^3(a\sqrt{3}-v)(u+v)^2$$

$$+\frac{u^3(a\sqrt{3}-u)(a\sqrt{3}-v)^3-v^3(a\sqrt{3}-u)^3(a\sqrt{3}-v)}{u+v}\Big]dudv$$

$$=(\tfrac{2}{3})^5\frac{1}{27\sqrt{3}a^6}\int_0^{\frac{1}{2}(a\sqrt{8})}\Big[8\sqrt{3}au^6-42a^2u^5+127\sqrt{3}a^3u^4-480a^4u^3+81\sqrt{3}a^5u^2$$

$$+216a^6u+16u^3(9a^4-u^4)\log\left(\frac{2u+a\sqrt{3}}{2u}\right)\Big]du=\frac{1507\sqrt{3}}{11664}a^2.$$

MISCELLANEOUS.

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104. Proposed by **HARRY S. VANDIVER**, Bala, Pa.

A Theorem of Fermat. The area of a right angled triangle with commensurable sides cannot be a square number. [Cf. Chrystal's *Algebra*, Vol. II., page 535.]

II. Solution by the PROPOSER.

Analytically expressed, Fermat's assertion is equivalent to the following: No positive integers can be found to satisfy the relation,

$$xy(x^2-y^2)=z^2 \ldots\ldots (1).$$

To prove this suppose that there are three integers, x_0, y_0, z_0, such that

$$x_0 y_0 (x_0^2-y_0^2)=z_0^2 \ldots\ldots (2).$$

Assume that x_0 and y_0 have a common factor, then we may put $x_0=pa$, and $y_0=pb$. Then substituting, $abp^4(a^2-b^2)=z_0^2$, or by division $ab(a^2-b^2)=\square=u^2$, say (where a and b are prime to each other). Hence it is sufficient to prove the impossibility of (2) only in the case when x_0 and y_0 are prime to each other.

The form ax_1^2 where $a=p\times q\times r\ldots\ldots$ (p, q, r, etc., being primes) and where either a or x_1 may become 1, represents all values of x_0. Putting it in this form, then the quantity a must occur as a factor in $y_0(x_0^2-y_0^2)$ since $x_0 y_0(x_0^2-y_0^2)$ is a square number, that is, one of the three following relations must be satisfied:

$$y_0 \equiv 0 (\bmod a)$$
$$x_0-y_0 \equiv 0 (\bmod a)$$
$$x_0+y_0 \equiv 0 (\bmod a)$$

Unless $a=1$, each of these relations show that x_0 and y_0 have a common factor, a result contrary to hypothesis.

Hence $x_0=x_1^2$, and it may be proved in a similar manner that $y_0=y_1^2$.

Substituting these values in (2) we have

$$x_1^2 y_1^2 (x_1^4-y_1^4)=z_0^2, \text{ whence } x_1^4-y_1^4=\square=z_1^2, \text{ say.}$$

Hence we infer, that if (1) is to be satisfied by integral values of x, y, and z it must be possible to find integers x_1, y_1, z_1 such that

$$x_1^4-y_1^4=z_1^2 \ldots\ldots (3).$$

But it may be shown that there are no such integers x_1, y_1, z_1.

For, consider the expression $x_1^2+y_1^2$. Its general form is bx_2^2 where b is the product of unequal primes, that is,

$$bx_2^2=x_1^2+y_1^2.$$

From this, $\quad x_1^2+y_1^2 \equiv 0 (\bmod b)$

and from (3), $\quad x_1^2-y_1^2 \equiv 0 (\bmod b)$

whence $\quad 2x_1^2 \equiv 0 (\bmod b)$

and $\quad 2y_1^2 \equiv 0 (\bmod b)$

and since x_1 and y_1 are prime to each other, we see that b must have either of the values 1 and 2; that is, either of the two following sets of relations must be satisfied:

$$1\text{st}\begin{cases} x_1^2+y_1^2=x_2^2 \\ x_1^2-y_1^2=y_1^2 \end{cases} \qquad 2\text{nd}\begin{cases} x_1^2+y_1^2=2x_2^2 \\ x_1^2-y_1^2=2y_2^2 \end{cases}$$

Take the first set, writing it as follows:

$$x_1^2-y_1^2=y_2^2 \ldots\ldots (4),$$
$$x_2^2-y_1^2=x_1^2 \ldots\ldots (5).$$

Then proceeding as we did with $x_1^4-y_1^4=z_1^2$, it can be shown that two of the following four sets must hold:

$$\text{I}\begin{cases} x_1+y_1=m^2 \\ x_1-y_1=n^2 \end{cases} \qquad \text{II}\begin{cases} x_2+y_1=r^2 \\ x_2-y_1=s^2 \end{cases} \quad (rs=x_1)$$

$$\text{III}\begin{cases} x_1+y_1=2m^2 \\ x_1-y_1=2n^2 \end{cases} \qquad \text{IV}\begin{cases} x_2+y_1=2r^2 \\ x_2-y_1=2s^2 \end{cases}$$

From (4) and (5) it will be seen that x_1 and x_2 are odd and y_1 is even. Therefore sets III and IV cannot be true. Using the first two and transforming

$$x_1=\frac{m^2+n^2}{2}=rs, \text{ or } y_1=\frac{m^2-n^2}{2}=\frac{r^2-s^2}{2}$$

$$\text{or } m^2+n^2=2rs \ldots\ldots (6),$$
$$m^2-n^2=r^2-s^2 \ldots\ldots (7).$$

Then it is *necessary* that there be integers a, b, l, k such that $r-s=ak$, $r+s=bl$ (any of the quantities a, b, l, k may be$=1$).

Then from (7), $m-n=ab$, $m+n=kl$.

Substituting in (6) the values found for r, s, m, and n from these relations, and reducing,

$$a^2b^2+k^2l^2+a^2k^2-b^2l^2=0,$$

whence $$a^2(b^2+k^2)=l^2(b^2-k^2),$$

and therefore $$b^4-k^4=\square \ldots\ldots (8).$$

Now since $x_1=\frac{m^2+n^2}{2}$, and also $ab=m-n$, and $\frac{m^2+n^2}{2}>m-n$,

we have $$x_1>ab,$$

whence $$x_1>b.$$

Hence, assuming $x_1^2+y_1^2=x_2^2$ and $x_1^2-y_1^2=y_2^2$ we have shown that if (3) is satisfied by x_1, y_1, z_1, then it follows that there is a relation $b^4-k^4=\square$ where $b<x_1$. Using the second assumption, and the only other possible one, namely,

$$\begin{cases} x_1^2+y_1^2=2x_2^2 \\ x_1^2-y_1^2=2y_2^2 \end{cases}$$

we obtain, by addition and subtraction,

$$x_2^2+y_2^2=x_1^2 \ldots\ldots(9),$$
$$x_2^2-y_2^2=y_1^2 \ldots\ldots(10),$$

and multiplying, $x_2^4-y_2^4=\square$. But (9) gives $x_2<x_1$.

Hence it has been shown that using this assumption, we can obtain from (3) a relation $x_2^4-y_2^4=\square$ such that $x_2<x_1$. Therefore, if (3) holds it is *always* possible to find a relation $x_2^4-y_2^4=z_2^2$ such that $x_2<x_1$.

From $x_2^4-y_2^4=z_2^2$ it is then possible to find $x_3^4-y_3^4=z_3^2$ where $x_3<x_2$, and so we can get any number of equations, of the type, $x_n^4-y_n^4=z_n^2$, where

$$x_n<_{n-1}<x_{n-2}\ldots\ldots<x_3<x_2<x_1.$$

Since x_n is always positive, by taking n sufficiently large, there is obtained $1-y_n^4=z_n^2$, which is impossible for positive integral values of y_n and z_n.

Hence the impossibility of $x_1^4-y_1^4=z_1^2$ has been completely established, and therefore the fact that $xy(x^2-y^2)=z^2$ cannot be satisfied, follows.

105. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

If the refractive index of a medium at any point be $\mu=x$, prove that the path of the ray will be the curve $\dfrac{2x}{a}=\dfrac{c}{a}\epsilon^{y/a}+\dfrac{a}{c}\epsilon^{-(y/a)}$, a and c being constants.

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy. Ohio University, Athens, O.

Let $(x,\ y)$ be any point in the path; $\mu=kx$, the index of refraction; and $p=dy/dx$. The differential equation to the path is

$$\frac{dp/dx}{1+p^2}=\frac{1}{\mu}\left(\frac{d\mu}{dy}-\frac{d\mu}{dx}\frac{dy}{dx}\right)\ldots\ldots(1).$$

$d\mu/dx=k$, and (1) reduces to

$$\frac{dp}{p(1+p^2)}=-\frac{dx}{x}\ldots\ldots(2).$$

Integrating, $\log\frac{px}{\surd(1+p^2)}=C=\log a$, say......(3).

This gives $p=\frac{dy}{dx}=\frac{a}{\surd(x^2-a^2)}$......(4).

Integrating (4), $y=a\log[x+\surd(x^2-a^2)]+C'$......(5).

Let $y=0$, when $x=b$; then $C'=-a\log[b+\surd(b^2-a^2)$, and (5) becomes after putting $c=b+\surd(b^2-a^2)$,

$$\frac{2x}{a}=\frac{c}{a}e^{y/a}+\frac{a}{c}e^{-(y/a)}......(6),$$

e being the Naperian base.

Also solved by *G. B. M. ZERR* and *L. C. WALKER*.

106. Proposed by J. W. YOUNG, Oliver Graduate Student in Mathematics, Cornell University, Ithaca, N. Y.

Prove that $\frac{(2m)!}{(m!)^2}$ is an integer; and more generally that $\frac{(nm)!}{(m!)^n}$ is an integer; m, n being any positive integers.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; and L. C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

It has been demonstrated that the product of any n successive integers is divisible by n!

$$\frac{(nm)!}{(m!)^n}\div\frac{[(n-1)m]!}{(m!)^{n-1}}=\frac{(nm)!}{[(n-1)m)!}\cdot\frac{1}{m!}=\frac{(nm)!}{(nm-m)!}\cdot\frac{1}{m!}$$

$$=\frac{nm(nm-1)(nm-2)......\text{to } m \text{ factors}}{m!}=\text{an integer.}$$

$\therefore$ If $\frac{[(n-1)m]!}{(m!)^{n-1}}$ is an integer, so is $\frac{(nm)!}{(m!)^n}$.

But $\frac{m!}{m!}$ is an integer. $\therefore$ $\frac{(2m)!}{(m!)^2}$ is an integer, and so on to $\frac{(nm)!}{(m!)^n}$.

Also solved by *H. S. VANDIVER*.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

156. Proposed by JAMES F. LAWRENCE, A. B., Professor of Mathematics, Rogers Academy, Rogers, Ark.

Suppose that in a meadow the grass is of uniform quality and growth, and that 6 oxen or 10 colts could eat up 3 acres of pasture in 18-25 of the time in which 10 oxen and 6 colts could eat up 8 acres; or that 600 sheep would require 2 6-7 weeks longer than 660 sheep to eat 9 acres. In what time could 1 ox, 1 colt, and 1 sheep eat up 1 acre of pasture, on the supposition that 588 sheep eat as much in a week as 6 oxen and 11 colts?

157. Proposed by B.F.FINKEL, A. M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

January 1, 1899, A and B entered into partnership for 3 years. A put in $10,000 and B put in $5,500. July 1, 1899, B put in $1,500 more. October 1, A took out $500. January 1, 1900, each put in $1,500. July 1, 1900, they dissolved partnership, and found that they had lost $846. What is each partner's share of the loss?

ALGEBRA.

159. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

If $x-1=3m$, $x^2-1=4n$, $x^3-1=5p$, where m, n, p are integers, find a general expression for x.

151. Proposed by J. SCHEFFER, A. M., Hagerstown. Md.

Represent the square root of $10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15}$ as the sum of three square roots.

152. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire. England.

If n quantities are made up of q sets of r each, find the number of permutations s at a time. It is supposed that the quantities in each set are alike, but different from those in the other sets.

GEOMETRY.

184. Proposed by ERWIN MARTIN, Principal of Schools, Mead, Neb.

If from any point in the circumference of a circle circumscribed about a triangle, perpendiculars are drawn to the sides, or the sides produced, of the inscribed triangle, the lines connecting the feet of the perpendiculars are collinear.

185. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

Given the tangential equations to two conics S, S', find the tangential co-ordinates of the join of the poles of two given parallel lines with respect to S. Deduce the tangential equation of the center of S, and find that of the intersection of S and S'.

186. Proposed by J. R. HITT, Professor of Mathematics, San Marcos, Tex.

If two sides of a triangle and its in-circle be given in position, the envelope of its circum-circle is a circle (*Mannheim*). [From Casey's *Sequel to Euclid.*]

CALCULUS.

150. Proposed by E. B. ESCOTT, Instructor in Mathematics, University of Michigan, Ann Arbor, Mich.

Find total area between the curve $x^4y-x^2+4y-1=0$ and the x-axis.

151. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Integrate the differential equation: $xy\dfrac{\partial^2 z}{\partial x.\,\partial y}=bx\dfrac{\partial z}{\partial x}+ag$.

MECHANICS.

140. Proposed by J. F. LAWRENCE, A. B., Professor of Mathematics, Rogers Academy, Rogers, Ark.

A long row of particles, each mass m, is placed on a smooth horizontal table. Each is connected with the two adjacent ones by similar light elastic strings of natural length l. They receive small longitudinal disturbances such that each of them proceeds to perform a harmonic oscillation. Prove that there will be two waves of vibration in opposite directions with the same velocity, viz, $l'\sqrt{\dfrac{E}{ml}}\,\dfrac{q}{\pi}\sin\dfrac{\pi}{q}$, when l' is the average distance between two successive particles, q the number of intervals between two particles in the same phase, and E the modulus of elasticity. [*Mathematical Tripos*, 1873.]

141. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

A simple pendulum hangs from a bicycle moving in a straight line. What deflection is produced by putting on the brake so as to exert on the machine a force equal to the nth of its weight?

DIOPHANTINE ANALYSIS.

101. Proposed by HARRY S. VANDIVER, Bala, Pa.

Prove that it is impossible to find integral values for x, y, and z such that the relation $x^2y+xz^2=y^2z$ is satisfied.

N. B. Problems in this department beginning with December number should be renumbered; those in December number being 96 and 97, respectively; January number, 98 and 99; February number, 100.

AVERAGE AND PROBABILITY.

125. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

A circle with unknown radius is described with its center at the extremity of the major axis of a given ellipse. Required the average area common to the circle and the ellipse.

126. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the average ellipse inscribed in a triangle, so that the sides of the triangle are tangent to the ellipse.

MISCELLANEOUS.

126. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

The declination of a certain fixed star is 12° 40′. Its altitude was observed one day to be 16° 40′. Three hours and twenty-four minutes later it was found to be 40° 20′. Find the latitude of the place of observation.

127. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Show how to calculate the velocity of an electrical discharge between a cloud and the earth. Is this velocity a function of the quantity of electricity discharged?

BOOKS AND PERIODICALS.

Elementary Treatise on Navigation and Nautical Astronomy. By Eugene L. Richards, M. A., Professor of Mathematics in Yale University. 16mo. Cloth, 173 pages. Price, $1.00. New York and Chicago: The American Book Co.

This is a neat little work dealing with some interesting problems in Navigation. It is elementary and clear in its treatment of subjects, and can therefore be easily read by any one having a knowledge of geometry and trigonometry.

Elements of Plane Geometry. By Alan Sanders, Hughes High School, Cincinnati, Ohio. 8vo. Cloth. Price, $1.00. New York and Chicago: The American Book Co.

This book does not differ very essentially from a great number of recent text-books on geometry. It contains a large list of well-selected exercises for original work on the part of the student, and the demonstrations and suggestions are good. The mechanical make-up of the book is first-class.

Vector Analysis. A Text-book for the Use of Students of Mathematical Physics. Founded upon the Lectures of J. Willard Gibbs, Ph. D., LL. D., Professor of Mathematical Physics in Yale University. By Edwin Bidwell Wilson, Ph. D., Instructor in Mathematics in Yale University. 8vo. Cloth, xviii+436 pages. Price, $4.00, net. New York: Charles Scribner's Sons.

This work belongs to that series of publications known as the "Yale Bicentennial Publications," and is the largest treatise thus far written on this subject in America. The body of the work is divided into six chapters of which the first treats of Addition and Scalar Multiplication; the second of Direct and Akew Products of Vectors; the third of Differential Calculus of Vectors; the fourth of Integral Calculus of Vectors; the fifth of Linear Vector Functions; and the sixth of Rotations and Strains; chapter seven deals with Miscellaneous Applications. This work will be of great value to the mathematician as well as the mathematical physicist.

An Elementary Book on Electricity and Magnetism and Their Applications. A text-book for Manual Training Schools and High Schools, and a Manual for Artesans, Apprentices, and Home Readers. By Dugal C. Jackson, C. E., Professor of Electrical Engineering, University of Wisconsin, member of the American Institute of Electrical Engineering, etc.; and John Price Jackson, M. E., Professor of Electrical Engineering, Pennsylvania State College, member of the American Institute of Electrical Engineering, etc. 16mo, cloth sides and leather back, xi+482 pages. New York: The Macmillan Co.

This is the best work of its kind that has yet appeared. It treats in a very lucid manner, every department of electricity and magnetism and their various applications. While written in somewhat untechnical language, yet it is clear, forceful, and scientifically accurate. In this work is found a practical treatment of the dynamo, motor, telegraph, telephone, electric lighting, electric smelting, welding, cooking, wireless telegraphy, Roentgen rays, and many other subjects. Here is a complete description of Niagara Falls Power Company's plant, with illustrations and a map, covering six pages. Throughout the book, frequent use of analogies are made, thus enabling the reader to obtain the clearest possible conceptions of electric and magnetic properties.

In the preparation of this book, the authors have contributed a most valuable work to the literature of the subject, and every one interested in the extension of knowledge and the enlarging of social conditions will want ta have a copy of the book.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single number, 25 cents. The Review of Reviews Co., 13 Astor Place, New York.

The Literary Digest. A Weekly Compendium of the Contemporaneous Thought of the World. Price, $3.00 per year in advance. Single number, 10 cents. Funk & Wagnalls Co., Publishers, 30 Lafayette Place, New York.

The Cosmopolitan. In International Illustrated Monthly Magazine. Edited and Published by John Brisben Walker. Price, $1.00 per year in advance. Single numbers, 10 cents. Irvington-on-the-Hudson.

ERRATA.

Page 34, first line under the title, for "1809" read 1896, 3rd edition; and for "Sir" read W.

Page 41, fourth line of second solution, for "rational" read real.

Page 41, ninth line of second solution, for "$\pm$" read $=$.

Page 43, in the last line but three of the first demonstration in Geometry, for "$AB=FB$" read $AD=FB$; and for "$\triangle ADK$" read $\triangle ADB$.

Page 47, first line, omit "Professor of Mathematics, Bowdoin College."

Page 54, thirteenth line of the solution, for "m" read m^2.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. IX. APRIL, 1902. No. 4.

THE MOTION OF A PROJECTILE IN A MEDIUM RESISTING AS THE CUBE OF THE VELOCITY.

By F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics, Defiance College, Defiance, Ohio.

For the motion of a projectile in a medium resisting as the *square* of the velocity, works on Mechanics, after assuming the fundamental equations of translatory motion,

$$\left|\begin{array}{l} \Sigma P\cos\alpha - \Sigma m(d^2x/dt^2)=0\ldots\ldots(\alpha), \\ \Sigma P\cos\beta - \Sigma m(d^2y/dt^2)=0\ldots\ldots(\beta), \\ \Sigma P\cos\gamma - \Sigma m(d^2z/dt^2)=0\ldots\ldots(\gamma), \end{array}\right| \ldots\ldots(1),$$

easily deduce the following equations:

$$\left|\begin{array}{l} M(d^2x/dt^2)=Mg\cos\alpha+Mcv^2\cos\alpha'\ldots\ldots(\alpha), \\ M(d^2y/dt^2)=Mg\cos\beta+Mcv^2\cos\beta'\ldots\ldots(\beta), \\ M(d^2z/dt^2)=Mg\cos\gamma+Mcv^2\cos\gamma'\ldots\ldots(\gamma), \end{array}\right| \ldots\ldots(2).$$

Let w be the *terminal* velocity of the projectile in the medium, and W the weight of the projectile in pounds; then the resistance of the air, when the velocity of the projectile is v, is equivalent to a force of $R=(v/w)^3W$ pounds. At the same time, the gravitational retardation produced is equivalent to a force $G=(v/w)^3g$ pounds.

Refer the motion of the projectile to oblique co-ordinate axes. Let the axis OX extend in the direction of projection at the point of infinite velocity, and the axis OY extend vertically downwards. In other words, cause the trajectory to lie in the vertical *co-ordinate-plane* XOY. This will enable us practically to *ignore* the co-ordinate axis OZ, and the equation (γ) in (1) and (2).

In the case under consideration, $\cos\alpha=0$, $\cos\beta=-1$, $\cos\gamma=0$, $\cos\alpha'=-(dx/ds)$, $\cos\beta'=-(dy/ds)$, $\cos\gamma'=(dz/ds)$, and $v=ds/dt$.

Transforming (2) under these conditions,

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -\frac{g}{w^3}\left(\frac{ds}{dt}\right)^3\frac{dx}{ds}\ldots\ldots(\alpha), \\ \frac{d^2y}{dt^2} &= -\frac{g}{w^3}\left(\frac{ds}{dt}\right)^3\frac{dy}{ds}+g\ldots\ldots(\beta), \\ \frac{d^2z}{dt^2} &= -\frac{g}{w^3}\left(\frac{ds}{dt}\right)^3\frac{dz}{ds}\ldots\ldots(\gamma), \end{aligned}\right|\ldots\ldots(3).$$

From (β) and (α) in (3),

$$\left(\frac{ds}{dt}\right)^3 = -\frac{w^3}{g}\left(\frac{d^2y}{dt^2}-g\right)\frac{ds}{dy} = -\frac{w^3}{g}\left(\frac{d^2x}{dt^2}\right)\frac{ds}{dx}\ldots\ldots(a).$$

Dividing (a) by ds/dt, etc.,

$$\left(\frac{d^2y}{dt^2}\right)\frac{dt}{dy} - \left(\frac{d^2x}{dt^2}\right)\frac{dt}{dx} = g\frac{dt}{dy}\ldots\ldots(b).$$

Multiplying (b) by $(dx/dt)(dy/dt)$, etc.,

$$\frac{d^2y}{dt^2}\cdot\frac{dx}{dt} - \frac{d^2x}{dt^2}\cdot\frac{dy}{dt} = g\frac{dx}{dt}\ldots\ldots(c).$$

$$\therefore\left(\frac{d^2y}{dt^2}\cdot\frac{dx}{dt} - \frac{d^2x}{dt^2}\cdot\frac{dy}{dt}\right)\frac{dt}{dx}\Big/\left(\frac{dx}{dt}\right)^2 = g\frac{dx}{dt}\Big/\left(\frac{dx}{dt}\right)^2\ldots\ldots(d).$$

$$\therefore\frac{d}{dt}\left(\frac{dy}{dt}\Big/\frac{dx}{dt}\right)\frac{dt}{dx} = g\left(1\Big/\frac{dx}{dt}\right)\ldots\ldots(e).$$

$$\therefore\frac{dp}{dt} = g\frac{dt}{dx},\ \text{or}\ \frac{dp}{dt}\cdot\frac{dx}{dt} = g\ldots\ldots(f),$$

in which p stands for dy/dx. *Apparently* (f) is *independent* of the law of resistance of the medium in the direction opposite to motion.

Assuming that the co-ordinate axis OX makes an angle ω with a horizontal plane, we have from the parallelogram of *velocity-components*

$$\cos(90-\omega)=\frac{(dy/dt)^2+(dx/dt)^2-(ds/dt)^2}{2(dy/dt)(dx/dt)} \ldots\ldots(g).$$

$$\therefore \left(\frac{ds}{dt}\right)^2=\left(\frac{dy}{dt}\right)^2-2\left(\frac{dy}{dt}\right)\left(\frac{dx}{dt}\right)\sin\omega+\left(\frac{dx}{dt}\right)^2$$

$$=\left(\frac{dx}{dt}\right)^2\left[\left(\frac{dy}{dx}\right)^2-2\left(\frac{dy}{dx}\right)\sin\omega+1\right]=\left(\frac{dx}{dt}\right)^2[p^2-2p\sin\omega+1]:\ldots\ldots(h).$$

From (a) in (3), by means of (h), may be deduced

$$\frac{d^2x}{dt^2}=-\frac{g}{w^3}\left(\frac{ds}{dt}\right)^2\left(\frac{ds}{dt}\times\frac{dx}{ds}\right)=-\frac{g}{w^3}\left(\frac{ds}{dt}\right)^2\frac{dx}{dt}$$

$$=-\frac{g}{w^3}\left(\frac{dx}{dt}\right)[p^2-2p\sin\omega+1]\ldots\ldots(i).$$

Eliminating g from (i), by means of (f),

$$\frac{dx^2}{dt^2}=-\frac{1}{w^3}\left(\frac{dp}{dt}\cdot\frac{dx}{dt}\right)\left(\frac{dx}{dt}\right)^3\left[\left(\frac{dy}{dx}\right)^2-2\left(\frac{dy}{dx}\right)\sin\omega+1\right]\ldots.(j).$$

$$\therefore w^2\left(\frac{dx}{dt}\right)^{-4}\frac{d^2x}{dt^2}=-\left[\left(\frac{dy}{dx}\right)^2-2\left(\frac{dy}{dx}\right)\sin\omega+1\right]\frac{d}{dt}\left(\frac{dy}{dx}\right)\ldots.(k).$$

Remembering that when $dy/dx=0$, $dx/dt=\infty$, and integrating (k),

$$\tfrac{1}{3}w^3(dx/dt)^{-3}=\tfrac{1}{3}p^3-p^2\sin\omega+p\ldots.(l).$$

$\therefore\ dx/dt=w/\sqrt[3]{(p^3-3p^2\sin\omega+3p)}\ldots.(m).$

Obvious transformations of (f) give

$$\frac{dp}{dt}=g\left(\frac{dx}{dt}\right)^{-1}, \text{ or } \frac{dp}{dx}\times\frac{dt}{dx}=g\left(\frac{dx}{dt}\right)^{-2}\ldots.(a').$$

$\therefore\ dp/dx=(g/w^2)[p^3-3p^2\sin\omega+3p]^{\frac{2}{3}}\ldots.(n).$

Taking reciprocals of (n), etc.,

$$(g/w^2)(dx/dp)=1/[p^3-3p^2\sin\omega+3p]^{\frac{2}{3}} \;....(o).$$

Multiplying the left-hand member of (o) by dy/dx and the right-hand member by p, etc.,

$$(g/w^2)(dy/dp)=p/[p^3-3p^2\sin\omega+3p]^{\frac{2}{3}} \;....(p).$$

From (f) and (m), can easily be deduced

$$\frac{dp}{dt}=g\Big/\frac{dx}{dt}=(g/w)[p^3-3p^2\sin\omega+3p]^{\frac{1}{3}} \;....(q).$$

Also, from (o), (p), and (q), respectively, by obvious integrations, etc.,

$$\frac{gx}{w^2}=\int_0^\infty d\left(\frac{dy}{dx}\right)\Big/\left[\left(\frac{dy}{dx}\right)^3-3\left(\frac{dy}{dx}\right)^2\sin\omega+3\left(\frac{dy}{dx}\right)\right]^{\frac{2}{3}} ....(r),$$

$$\frac{gy}{w^2}=\int_0^\infty \frac{dy}{dx}d\left(\frac{dy}{dx}\right)\Big/\left(\frac{dy}{dx}\right)^3-3\left(\frac{dy}{dx}\right)^2\sin\omega+3\left(\frac{dy}{dx}\right)\Big]^{\frac{2}{3}} ....(s),$$

$$\frac{gt}{w}=\int_0^\infty d\left(\frac{dy}{dx}\right)\Big/\left[\left(\frac{dy}{dx}\right)^3-3\left(\frac{dy}{dx}\right)^2\sin\omega+3\left(\frac{dy}{dx}\right)\right]^{\frac{1}{3}} ....(t).$$

Assume $z=M^2\sqrt[3]{(p^3-3p^2\sin\omega+3p)/p}....(\beta')$.

Cubing (β'), differentiating, etc.,

$$6z^2dz=6M^6[(p\sin\omega-2)/p^3]dp....(\gamma').$$

From (β'), after cubing, can be formed the expression,

$$4z^3-g_3=[(4M^6-g_3)p^2-12M^6p\sin\omega+12M^6]/p^2....(\delta'),$$

which is a perfect square when we have $4M^6-g_3=3M^6\sin^2\omega$, or when $g_3=M^6(4-3\sin^2\omega)$. Substituting this value of g_3 in (δ), etc.,

$$\sqrt{(4z^3-g_3}=M^3(p\sin\omega-2)\sqrt{(3)}/p....(\epsilon').$$

Dividing (γ') by (ϵ'), etc., also making the *arbitrary* constant factor $M^2=\frac{1}{3}$, we have

$$\frac{6z^2dz}{\sqrt{(4z^3-g_3)}}=\frac{6M^6(p\sin\omega-2)dp}{p^3}\times\frac{p}{M^3(p\sin\omega-2)\sqrt{3}}....(\varsigma').$$

$$\therefore \frac{dz}{\sqrt{(4z^3-g_3)}}=\frac{M^3\sqrt{(3)}}{3p^2z^2}\times\frac{g[p^3-3p^2\sin\omega+3p]^{\frac{2}{3}}\,dx}{w^2}$$

$$=\frac{M^3\sqrt{(3)}\times p^2}{3p^2\times M^4[p^3-3p^2\sin\omega+3p]^{\frac{2}{3}}}\times\frac{g[p^3-3p^2\sin\omega+3p]^{\frac{2}{3}}\,dx}{w^2}\ldots.(\eta').$$

$$\therefore \frac{gx}{w^2}=\int_0^\infty\frac{dz}{\sqrt{(4z^3-g_3)}},=*(p,\,u)^{-1}(z;\,0,\,g_3)\ldots.(\theta').$$

That is, $z=*(p,\,u)\left(\frac{gx}{w^2};\,0,\,g_3\right)\ldots.(\iota')$,

which is an *incomplete* form of the *first* of the three Weierstrassian functions by means of which it is possible to express any elliptic integral; and this function is expressed in its *inverted* form according to the Abelian spstem of notation. Now, $*(p,\,u)'\left(\frac{gx}{w^2}\right)=\frac{p\sin\omega-2}{3p}$; and if $z=a$ at the vertical asymptote, where $p=\infty$, we must have

$$*(p,\,u)'\left(\frac{ga}{w^2}\right)=\frac{\sin\omega}{3},\text{ and }*(p,\,u)\left(\frac{ga}{w^2}\right)=\tfrac{1}{3}.$$

$$\therefore *(p,\,u)'\left(\frac{ga}{w^2}\right)-*(p,\,u)'\left(\frac{gx}{w^2}\right)=\frac{2}{3p}\ldots.(\kappa').$$

From (κ') by obvious transformations

$$p=\frac{dy}{dx}=\tfrac{2}{3}/\left[*(p,\,u)'\left(\frac{ga}{w^2}\right)-*(p,\,u)'\left(\frac{gx}{w^2}\right)\right]\ldots.(\lambda').$$

$$\therefore y=\int_0\left[6*(p,\,u)^2\left(\frac{ga}{w^2}\right)/\left[*(p,\,u)'\left(\frac{ga}{w^2}\right)-*(p\text{-}\,u)'\left(\frac{gx}{w^2}\right)\right]\right]dx\ldots.(\mu'),$$

which is the equation of the trajectory expressed according to the Weierstrassian system of functional notation.

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*Being without the proper character for the $(p,\,u)$ function we designate it by $(p,\,u)$.

APPLICATIONS OF A THEOREM REGARDING CIRCULANTS.

By HARRY S. VANDIVER, Bala, Pa.

A circulant of the nth order is a simple determinant of the following type:

$$\begin{vmatrix} a_1 & \omega_n a_2 & \omega_n^2 a_3 & \dots & \omega_n^{n-1} a_n \\ \omega_n^{n-1} a_n & a_1 & \omega_n a_2 & \dots & \omega_n^{n-2} a_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ \omega_n a_2 & \omega_n^2 a_3 & \omega_n^3 a_4 & \dots & a_1 \end{vmatrix}$$

where ω_n is a special nth root of unity. This is the product of n linear factors, as follows:

$$\begin{array}{l} a_1 + \omega_n a_2 + \omega_n^2 a_3 + \omega_n^3 a_4 \dots \omega_n^{n-1} a_n \\ a_1 + \omega_n^2 a_2 + \omega_n^4 a_3 + \omega_n^6 a_4 \dots \omega_n^{n-2} a_n \\ a_1 + \omega_n^3 a_2 + \omega_n^6 a_3 + \omega_n^9 a_4 \dots \omega_n^{n-3} a_n \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_1 + a_2 + a_3 \dots \dots a_n \end{array}$$

A theorem regarding such functions is:

The product of any number of circulants each of the order n can be expressed as a circulant of the nth order.

For simplicity, we will prove this for continuants of the third order; the procedure is precisely the same when $n>3$. We have:

$$\begin{vmatrix} a_1 & \omega_3 a_2 & \omega_3^2 a_3 \\ \omega_3^2 a_3 & a_1 & \omega_3 a_2 \\ \omega_3 b_2 & \omega_3^2 a_3 & a_1 \end{vmatrix} \times \begin{vmatrix} b_1 & \omega_3 b_2 & \omega_3^2 b_3 \\ \omega_3^2 b_3 & b_1 & \omega_3 b_2 \\ \omega_3 b_2 & \omega_3^2 b_3 & b_1 \end{vmatrix}$$

This equals the product of the six factors:

$$\begin{array}{l} (a_1 + a_2 + a_3)(b_1 + b_2 + b_3) \\ (a_1 + \omega_3 a_2 + \omega_3^2 a_2)(b_1 + \omega_3 b_2 + \omega_3^2 b_3) \\ (a_1 + \omega_3^2 a_2 + \omega_3 a_3)(b_1 + \omega_3^2 b_2 + \omega_3 b_3) \end{array}$$

Taken two and two, these give:

$$\begin{array}{l} A_1 + A_2 + A_3 \\ A_1 + \omega_3 A_2 + w_3^2 A_3 \\ A_1 + \omega_3^2 A_2 + \omega_3 A_3. \end{array}$$ Whence the theorem.

If it is true for two circulants it is true for any number.

Corollary. Any integral power of a circulant of the nth order is a circulant of the same order.

Some of the applications of this theorem and corollary to Diophantine analysis follow:

$$\begin{vmatrix} x & iy \\ iy & x \end{vmatrix} \times \begin{vmatrix} a & ib \\ ib & a \end{vmatrix} = \begin{vmatrix} c & id \\ id & c \end{vmatrix}$$

or $(x^2+y^2)(a^2+b^2)=c^2+d^2$, a well known theorem.

If the relation $x^2+bxy+cy^2=1$ possesses one solution in integers it possesses an infinite nnmber.

The function $x^2+bxy+cy^2$ can be expressed as a circulant as follows:

$$\begin{vmatrix} \dfrac{2x+by}{2} & \dfrac{\sqrt{(b^2-ac)}}{2}y \\ \dfrac{\sqrt{(b^2-4ac)}}{2}y & \dfrac{2x+by}{2} \end{vmatrix}$$

Suppose the equation possesses one solution, say $x=m$, $y=n$.

Then $m^2+bmn+cn^2=1$; then we can put

$$(m^2+bmn+cn^2)^k=x^2+bxy+cy^2.$$

Putting $k=2$, and decomposing both circulants into their irrational factors, there is obtained:

$$\frac{2x+by-y\sqrt{(b^2-4c)}}{2}=\left[\frac{2m+bn-n\sqrt{(b^2-4c)}}{2}\right]^2$$

and

$$\frac{2x+by+y\sqrt{(b^2-4c)}}{2}=\left[\frac{2m+bn+n\sqrt{(b^2-4c)}}{2}\right]^2$$

Taking the first relation and expanding

$$4x+2by-2y\sqrt{(b^2-4c)}=(2m+bn)^2-2n(2m-bn)\sqrt{(b^2-4c)}+n^2(b^2-4c).$$

Equating rational and irrational parts

$$4x+2by=(2m+bn)^2+n^2(b^2-4c) \text{ and } y=n(2m+bn).$$

By substitution we find $x=m^2-cn^2$.

Proceeding in the same manner when $k>2$, we find integral values for x and y. Hence since k is unlimited the number of solutions is unlimited. Hence the theorem.

To show that, for every value of n, the relation

$$x^3+ay^3+a^2z^3-3axyz=v^n$$

has (*when a is an integer*) *an infinite number of integral solutions in x, y, and z.*

The function $x^3+ay^3+a^2z^3-3axyz$ expressed as a circulant is:

$$\begin{vmatrix} x & y\sqrt[3]{a} & z\sqrt[3]{a^2} \\ z\sqrt[3]{a^2} & x & y\sqrt[3]{a} \\ y\sqrt[3]{a} & z\sqrt[3]{a^2} & x \end{vmatrix}$$

We then can assume $(x_1^3+ay_1^3+a^2z_1^3-3ax_1y_1z_1)^n=x^3+ay^3+a^2z^3-3axyz$.

Put $n=2$. Then

$$(x_1+y_1\sqrt[3]{a}+z_1\sqrt[3]{a^2})^2=x+y\sqrt[3]{a}+z\sqrt[3]{a^3}.$$

From which, by expanding and equating irrationals,

$$x=x_1^2+2ay_1z_1$$
$$y=z_1^2a+2x_1y_1$$
$$z=y_1^2+2x_1y_1.$$

The number of values that can be assigned to x_1, y_1, z_1 is infinite; hence the theorem is proved for $n=2$. When $n>2$ it may be proved by proceeding in the same manner.

In Article 9, entitled, "Of the manner of finding Algebraic Functions of all degrees, which when multiplied together may always produce similar functions" in the Additions to Euler's Algebra, Lagrange gives developments which may be considered as applications of the circulant theorem.

Bala, Pa., July 13, 1901.

THE BETWEENNESS ASSUMPTIONS.

By DR. GEORGE BRUCE HALSTED.

In his "Vorlesung ueber Euklidische Geometrie" (Wintersemester 1898 99), Hilbert gave the most remarkable set of axioms ever created for the founding of geometry. These were published in his Festschrift, 1899, treated of in my "Supplementary Report" to the A. A. A. S., (*Science*, Nov. 8, 1901). They have been given in English, though in part erroneously, in D. E. Smith's "The Teaching of Elementary Mathematics," (Macmillan).

The first group of these assumptions Hilbert calls *Axioms of Association.* Of these we need only mention—

I 1. Two distinct points, A, B, always determine a straight a.

I 7. On every straight there are at least two points; in every plane there are at least three non-co-straight points; and in space there are at least four non-co-straight, non-co-planar points.

The second group of assumptions Hilbert calls *Axioms of Arrangement.*

It is these that I call the *Betweenness Assumptions.*

Of them Hilbert says in §3: "The axioms of this group define the idea "between," and make possible on the basis of this idea the *arrangement* of the points on a straight, in a plane and in space.

Convention. The points of a straight stand in certain relations to one another, to describe which especially the word "*between*" serves us.

II 1. If A, B, C are points of a straight, and B lies between A and C, then B also lies between C and A.

II 2. If A and C are two points of a straight, then there is always at least one point B, which lies between A and C, and at least one point, such that C lies between A and D.

II 3. Of any three points of a straight there is always one and only one, which lies between the other two.

II 4. Any four points A, B, C, D of a straight can always be so arranged that B lies between A and C and also between A and D, and furthermore C lies between A and D and also between B and D.

DEFINITION. The system of two points A and B, which lie upon a straight a, we call a *sect*, and designate it with AB or BA.

The points between A and B are said to be points of the sect AB or also situated *within* the sect AB; all remaining points of the straight a are said to be situated *without* the sect AB. The points A, B are called *endpoints* of the sect AB.

II 5. Let A, B, C be three points not co-straight and a a straight in the plane ABC striking none of the points A, B, C: if then the straight a goes through a point within the sect AB, it must always go either through a point of the sect BC or through a point of the sect AC."

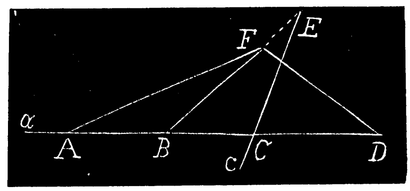
A year ago, while preparing my Report, I discussed with my pupil, R. L. Moore, the interpretation of the term "angeordnet" in II 4, and whether II 4 might not be demonstrated from the other assumptions.

Finally I wrote to Hilbert asking if he recognized the desirability of any change, as I wished to use his axioms as basis for a text-book on geometry.

His answer bears date April 2, 1902, and was received April 14, 1902. It begins: "Ueber Ihre Idea aus meinen Grundlagen eine Schul-Geometrie zu machen, bin ich sehr erfrunt." Finally, at the very end of the letter, in a postscript, Hilbert says; "Instead of II 4 I believe it suffices simply to say: If B lies between A and C and C between A and D, then lies also B between A and D; and then to prove my old II 4 as theorem."

I read this to Mr. Moore, and suggested his filling in the proof. Early next morning he announced to me that he had demonstrated Hilbert's new axiom, thus totally eliminating II 4, and reducing the Betweenness Assumptions from five to four. Mr. Moore has no intimation that any one has ever tried to prove these theorems. He started simply from the fact that two weeks ago Hilbert thought an assumption necessary of which now the demonstration, as I have written it out from his oral communication, seems of most unexpected simplicity and elegance.

THEOREM I. If B is between A and C, and C is between A and D, then C is between B and D.



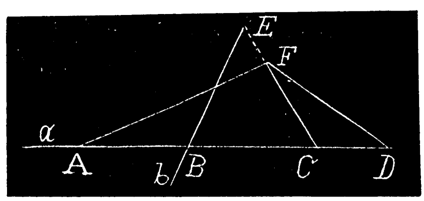
PROOF. Let A, B, C, D be on a. Through C take a straight c other than a. On c take a point E other than C. On the straight BE, between B and E take F.

Thus between B and F is no point of c. Now between A and F there can be no point of c, else c would (by II 5) have a point between A and B, since by the construction of F, c cannot have point between B and F. Thus C would be between A and B, contrary to our hypothesis that B is between A and C.

Thus, since c cannot have a point between A and F, it must (by II 5) have a point between F and D. So we have the three non-co-straight points F, B, D, and c with a point between F and D, and, by construction, none between F and B. Therefore it must (by II 5) have a point between B and D. So C is between B and D.

THEOREM II. [Hilbert's new axiom, II 4′]. If B is between A and C, and C is between A and D, then B is between A and D.

PROOF. Let A, B, C, D be on a. Through B take a straight b other than a. On b take a point E other than B. On the straight CE, between C and E take F. Thus between C and F is no point of b. Then since by hypothesis B is between A and C, therefore b must (by II 5) have a point between A and F. Thus we have the three non-co-straight points A, F, D, and b with a point between A and F. Therefore b must (by II 5) have a point between A and D or between F and D. But it cannot have a point between F and D, for then it must (by II 5) have a point either between F and C, contrary to our construction, or else between C and D, contrary to Theorem I, by which C is between B and D. Therefore it has a point between A and D. So B is between A and D.



THEOREM III. [Hilbert's old axiom II 4]. Any four points of a straight can always be so lettered $ABCD$, that B is between A and C and also between A and D, and furthermore C is between A and D and also between B and D.

PROOF. We may (by II 3) letter three of our points B, C, D, with C be-

tween B and D. Now as regards B and D, and our fourth point A, either A is between B and D, or B is between A and D, or D is between A and B.

If B is between A and D, we have fulfilled the hypothesis of Theorems I and II. If D is between A and B, then interchanging the lettering for B and D, that is, calling B, D, and D, B, we have again fulfilled the hypothesis of Theorems I and II.

There only remains to consider the case where A is between B and D. If now C is between D and A we have fulfilled the hypothesis of Theorems I and II, by calling D, A, and C, B, and A, C, and B, D. If however A were between C and D we would have fulfilled the hypothesis of Theorems I and II by writing for A, B, for D, A, and for B, D.

We have only left one case to consider, that where D is between A and C. This case is impossible. Suppose $ABCD$ on a. Through C take a straight c other than a. On c take a point E other than C. On the straight DE between D and E take F. Thus between D and F is no point of c.

Then since by hypothesis C is between B and D, therefore c must (by II 5) have a point between B and F. Therefore we have the three non-co-straight points B, F, A, and c with a point between B and F. Therefore c has (by II 5) a point between B and A or a point between F and A.

But it cannot have a point between F and A, else it would (by II 5) have a point between F and D, contrary to our construction, or else between D and and A, giving C between D and A, contrary to our hypothesis D between A and C. So C would be between B and A, and D between A and C, and therefore (by Theorem II) D between A and B, contrary to our hypotheris A between B and D.

Thus there is always such a lettering that B is between A and C, and C between A and D, whence (by Theorem I) C is between B and D, and (by Theorem II) B is between A and D.

Austin, Texas, April 17, 1902.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

155. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

A bought a horse, which he sold to B at a *loss* of $m=6\%$; B sold the horse to C at a *loss* of $n=6\%$; and C sold the horse to D at a *gain* of $p=12\frac{1}{2}\%$. How much did A *lose*, if C gained $\$G=\$26.79$?

Solution by J. R. HITT, Coronal Institute, San Marcos, Tex.

Let 100%=what horse cost A. Then $\dfrac{(100-6)(100-5)\times 12\frac{1}{2}}{(100)^2}$ $=94\times(.95)(.12\frac{1}{2})=.111625$ of what horse cost A=C's gain=\$26.79.

Hence A's loss $=\dfrac{\$26.79\times.06}{.111625}=\$14.40$.

Also solved by *G. B. M. ZERR.*

ALGEBRA.

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136. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Solve $a^{x^2}b^{y^2}=c....(1)$, and $c^{x+y}=ab....(2)$.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and J. SCHEFFER, A. M., Hagerstown, Md.

Let $\log a=m$, $\log b=n$, $\log c=p$. Then $mx^2+ny^2=p$, and $px+py=m+n$. From which we easily get

$$x=\frac{n}{p}\pm\frac{1}{p}\sqrt{\frac{p^3-mn(m+n)}{m+n}},\quad y=\frac{m}{p}\mp\frac{1}{p}\sqrt{\frac{p^3-mn(m+n)}{m+n}}.$$

Solved in a similar manner by *H. C. WHITAKER*, and *L. C. WALKER.*

137. Proposed by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

Solve, if possible, $a^x+b^x=c$.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College. Philadelphia, Pa.; LON C. WALKER, A. M., Petaluma High School, Petaluma, Cal.; and F. P. MATZ, Sc. D., Ph. D., Defiance College. Defiance. Ohio.

Let $\log a=m$, $\log b=n$. Then

$$a^x=1+mx+\frac{m^2x^2}{2!}+\frac{m^3x^3}{3!}+\frac{m^4x^4}{4!}+....$$

$$b^x=1+nx+\frac{n^2x^2}{2!}+\frac{n^3x^3}{3!}+\frac{n^4x^4}{4!}+....$$

Adding, $c-2=(m+n)x+\dfrac{m^2+n^2}{2!}x^2+\dfrac{m^3+n^3}{3!}x^3+\dfrac{m^4+n^4}{4!}x^4....$

By reversion of series,

$$x=\frac{c-2}{m+n}-\frac{(m^2+n^2)(c-2)^2}{(m+n)^3\,2!}+\frac{[3(m^2+n^2)^2-(m+n)(m^3+n^3)](c-2)^3}{(m+n)^5\,3!}$$

$$-\frac{[15(m^2+n^2)^2-10(m+n)(m^2+n^2)(m^3+n^3)+(m+n)^2(m^4+n^4)](c-2)^4}{(m+n)^7\,4!}+..$$

Also solved by *WM. E. HEAL*, and *J. SCHEFFER.*

138. Proposed by HARRY S. VANDIVER. Bala, Pa.

Show that the number of solutions in positive integers for x, y, and z of $x^3+2y^3+4z^3-6xyz=1$ is infinite.

Solution by the PROPOSER.

Suppose, for the present, that there is one solution of the proposed equation. Let $x=a$, $y=b$, $z=c$, satisfy it. Then

$$a^3+2b^3+4c^3-6abc=1. \quad \text{Whence } [a^3+2b^3+4c^3-6abc]^n=1.$$

Then whatever values of x, y, and z we take consistent with $x^3+2y^3+4z^3-6xyz=1$, we will have

$$[a^3+2b^3+4c^3-6abc]^n=x^3+2y^3+4z^3-6xyz,$$

which is satisfied by the following assumptions:

$$\begin{aligned} x+y\sqrt[3]{2}+z\sqrt[3]{4}&=[a+b\sqrt[3]{2}+c\sqrt[3]{4}]^n \\ x+\omega y\sqrt[3]{2}+\omega^2 z\sqrt[3]{4}&=[a+\omega b\sqrt[3]{2}+\omega^2 c\sqrt[3]{4}]^n \quad (1) \\ x+\omega^2 y\sqrt[3]{2}+\omega z\sqrt[3]{4}&=[a+\omega^2 b\sqrt[3]{2}+\omega c\sqrt[3]{4}]^n \end{aligned}$$

(where $\omega^3=1$) as may be seen by multiplying together the equations, term for term. (1) gives on expanding

$$x+y\sqrt[3]{2}+z\sqrt[3]{4}=A+B\sqrt[3]{2}+C\sqrt[3]{4}$$

where A, B, and C are positive integers.

Hence, we must have $x=A$, $y=B$, $z=C$.

(2) and (3) give likewise

$$\begin{aligned} x+\omega y\sqrt[3]{2}+\omega^2 z\sqrt[3]{4}&=A+\omega B\sqrt[3]{2}+\omega^2 C\sqrt[3]{4} \\ x+\omega^2 y\sqrt[3]{2}+\omega z\sqrt[3]{4}&=A+\omega^2 B\sqrt[3]{2}+\omega C\sqrt[3]{4} \end{aligned}$$

In each of these we will have as in (1) $x=A$, $y=B$, $z=C$.

Therefore, x, y, and z have an infinite number of integral values depending on the value of n; provided, as we assumed at the start, that they have one set of values.

This one set of values is $x=1$, $y=1$, $z=1$; for $1^3+2\times3^3+4\times1^3-6=1$.

Let us find, for instance, the values of x, y, and z for $n=2$, having given that $a=1$, $b=1$, $c=1$, we have

$$x+y\sqrt[3]{2}+z\sqrt[3]{4}=[1+\sqrt[3]{2}+\sqrt[3]{4}]^2, \text{ or } x=5,\ y=4,\ z=3,$$

which satisfy. Putting $n=3$, we find another set, and so on.

The method used in the solution of this problem is a particular case of a general principle applicable to the solution of many Diophantine questions.

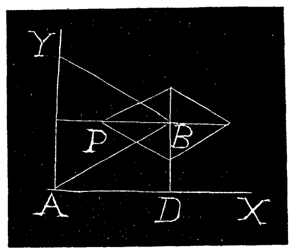
GEOMETRY.

167. Proposed by JOHN M. QUINN, Professor of Mathematics. High School, Warren, Pa.

If at the vertex of an isosceles triangle one of whose basal vertices is pivoted and the other free to move in a straight line, a rhombus be pivoted with sides parallel to the sides of the triangle, the locus of every point on the rhombus except the one which is its intersection with the fixed side of the triangle is an ellipse.

Solution by the PROPOSER.

Notation. Let X and Y be rectangular axes; θ the angle BAX; m equal segments of the sides of the triangle and the rhombus; $AB=a$, $BD=y$, and $AD=x$, and YBA an isosceles triangle with one basal vertex pivoted at A. The other basal vertex is free to move along the line AY.



To prove that the locus of any point as P is an ellipse.

PROOF. $y/a=\sin\theta$. $\therefore$ $y^2/a^2=\sin^2\theta$.

$x=a\cos\theta-2m\cos\theta=\cos\theta(a-2m)$.

$$\frac{x}{a-2m}=\cos\theta,\ \frac{x^2}{(a-2m)^2}=\cos^2\theta.$$

$$\therefore \frac{y^2}{a^2}+\frac{x^2}{(a-2m)^2}=\sin^2\theta+\cos^2\theta=1.$$ $\therefore$ the locus of p is an ellipse.

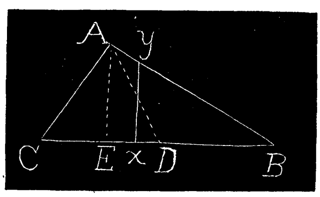
Similarly for any other point *mutatis mutandis*. Q. E. D.

168. Proposed by MISS GUBELMAN, Student Southern Illinois State University, Carbondale, Ill.

To draw a perpendicular to one side of a triangle dividing it into two equivalent parts.

Solution by the PROPOSER.

1. Let ABC be the triangle. Draw the median AD and the perpendicular AE. Construct BX such that $BX^2=BD.BE$. Draw the perpendicular XY.



$\therefore$ $\triangle BXY:\triangle BAE=BX^2:BE^2$.

Similar triangles$=BD\times BE=BE^2=BD:BE$.

Also $\triangle BAD:\triangle BAE=BD:BE$, having equal altitudes.

$\therefore$ $\triangle BXY:\triangle BAE=\triangle BAD:\triangle BAE$.

$\therefore$ $\triangle BXY=\triangle BAD$. But $\triangle BAD=\frac{1}{2}\triangle ABC$. Median.

$\therefore$ $\triangle BXY=\frac{1}{2}\triangle ABC$. $\therefore$ XY is the required perpendicular.

Also solved by *G. B. M. ZERR, DANIEL B. NORTHRUP, L. C. WALKER, J. SCHEFFER, H. C. WHITAKER, H. B. PENHOLLOW*, and *P. W. WEBBER*.

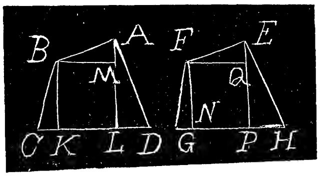
169. Proposed by S. F. NORRIS, Professor of Astronomy and Mathematics, Baltimore City College, Baltimore, Md.

Theorem. Two quadrilaterals having three sides of the one equal to the thre corresponding sides of the other, each to each, and the two corresponding angles adjacent to the unknown sides equal, each to each, are equal figures. [From *Olney's Geometry*, Section VIII, Proposition XIV].

1. Required proof. 2, Is this proposition found in any other text-book of Geometry?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let $ABCD$, $EFGH$ be the two quadrilaterals; $BC=FG, AB=EF, AD=EH$, $\angle BCD=\angle FGH$, $\angle ADC=\angle EHG$. Draw BK, AL perpendicular to CD; FN, EP perpendicular to GH; BM perpendicular to AL; FQ perpendicular to EP.



Right triangles BCK and FGN are equal, also right triangles ALD and EPH, having hypotenuse and acute angle of one equal to hypotenuse and acute angle of other.

$\therefore$ $BK=FN$, $AL=EP$, also $AL-BK=AM=EP-FN=EQ$.

$\therefore$ right triangles ABM=right triangle FEQ; since $AB=FE$ and $AM=EQ$,

$\therefore$ $BM=FQ$. $\therefore$ $BM=KL=FQ=NP$. $\therefore$ $BKLM=FNPQ$.

$\therefore$ $BCK+ADL+ABM+BKLM=FGN+EPH+FEQ+FNPQ$.

$\therefore$ $ABCD=EFGH$.

Also solved by *J. SCHEFFER*.

170. Proposed by CHARLES C. CROSS, Whaleyville, Va.

If p, q, r are the distances of the orthocenter from the sides, prove that

$$4\left[\frac{a}{p}+\frac{b}{q}+\frac{c}{r}\right]=\left[\frac{a}{p}+\frac{b}{q}-\frac{c}{r}\right]\left[\frac{a}{p}-\frac{b}{q}+\frac{c}{r}\right]\left[-\frac{a}{p}+\frac{b}{q}+\frac{c}{r}\right].$$

Solution by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

From well known theorems we have

$$\begin{array}{lcl} a=2R\sin A & & p=2R\cos B\cos C \\ b=2R\sin B & \text{and} & q=2R\cos C\cos A \\ c=2R\sin C & & r=2R\cos A\cos B \end{array}$$

Whence $a/p=\tan B+\tan C$, $b/q=\tan C\tan A$, $c/r=\tan A+\tan B$.

Therefore, $a/p+b/q+c/r=2(\tan A+\tan B+\tan C)$.

$$-a/p+b/q+c/r=2\tan A$$
$$a/p-b/q+c/r=2\tan B$$
$$a/p+b/q-c/r=2\tan C.$$

Since $\tan A+\tan B+\tan C=\tan A\tan B\tan C$, the theorem is proved.

Also demonstrated by *L. C. WALKER*, *J. SCHEFFER*, *H. C. WHITAKER*, *G. B. M. ZERR*, and the *PROPOSER*.

171. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the nearest distance of the parabola $y^2=16x$ and the ellipse $16x^2+9y^2-160-144y+832=0$.

Solution by the PROPOSER.

Let the equation of the parabola be $y^2=4px$, and that of the ellipse

$$\frac{(y-\beta)^2}{b^2}+\frac{(x-\alpha)^2}{a^2}=1.$$

A normal to the ellipse expressed by the tangent of the angle it makes with the axis of x has the equation

$$y-\beta=m(x-\alpha)-\frac{(a^2-b^2)m}{\sqrt{(b^2m^2+a^2)}},$$

and in the case of the parabola $y=mx-2pm-pm^3$. Since the shortest distance is measured off on a common normal, the two equations should be identical, and therefore

$$\beta-m\alpha-\frac{(a^2-b^2)m}{\sqrt{(b^2m^2+a^2)}}=-2pm-pm^3.$$

From this equation m is to be found. It leads to an equation of the 8th degree, which for numerical values presents no difficulty.

Substituting now the equation of the normal in both the equations of the parabola and ellipse, we find the co-ordinates of the points of intersection. Denoting these by x', y', and x'', y'', we find for the shortest distance the expression $\sqrt{[(x'-x'')^2+(y'-y'')]^2}$.

The numerical equations given having a common value, x lying between 6 and 7, and intersecting, therefore present no suitable example.

Also solved by *G. B. M. ZERR.*

CALCULUS.

129. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Among all quadrilaterals inscribed in an ellipse, to determine that which contains the greatest area.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Walton furnishes a solution of this interesting problem, which for its elegance and simplicity I reproduce here.

Let the equation of the ellipse be $x^2/a^2+y^2/b^2=1$, and let the angular points of the quadrilateral be (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4). Then, u denoting the area of the quadrilateral,

$$2u=x_2y_1-x_1y_2+x_3y_2-x_2y_3+x_4y_3-x_3y_4+x_1y_4-x_4y_1.$$

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1,\quad \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}=1,\quad \frac{x_3^2}{a^2}+\frac{y_3^2}{b^2}=1,\quad \frac{x_4^2}{a^2}+\frac{y_4^2}{b^2}=1.$$

In order that u may be a maximum, we have, differentiating the last four equations, putting $du=0$, and using the method of indeterminate multipliers,

$$\frac{\lambda_1 x_1}{a^2}+y_4-y_2=0,\quad \frac{\lambda_1 y_1}{b^2}-x_4+x_2=0.$$

. . . . . .

Hence we have $\frac{x_1}{a^2}(x_4-x_2)=-\frac{y_1}{b^2}(y_4-y_2)\ldots.(1)$,

$$\frac{x_2}{a^2}(x_1-x_3)=-\frac{y_2}{b^2}(y_1-y_3)\ldots.(2),\quad \frac{x_3}{a^2}(x_2-x_4)=-\frac{y_3}{b^2}(y_1-y_4)\ldots.(3),$$

$$\frac{x_4}{a^2}(x_3-x_1)=-\frac{y_4}{b^2}(y_3-y_1)\ldots.(4).$$

From (1) and (3) we have $\frac{y_3}{x_3}=\frac{y_1}{x_1}\ldots.(5)$, and from (2) and (4), $\frac{y_4}{x_4}=\frac{y_2}{x_2}\ldots.(6)$. Also from (1) and (2), $\frac{1}{a^2}(x_1x_4-x_2x_3)=-\frac{1}{b^2}(y_1y_4-y_2y_3)$.

From the last three equations we see that

$$\frac{1}{a^2}(x_1x_4-x_2x_3)=-\frac{1}{b^2}y_1y_2\left[\frac{x_4}{x_2}-\frac{x_3}{x_1}\right],\text{ and therefore }\frac{y_1y_2}{x_1x_2}=-\frac{b^2}{a^2}..(7).$$

The equations (5) and (6) show that the diagonals of the quadrilaterals are diameters of the ellipse, and (7) shows that they are conjugate diameters.

Also solved by *G. B. M. ZERR*, *A. H. HOLMES*, and *L. C. WALKER*.

130. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Solve the differential equation $x^x(\frac{dy}{dx}+y\log x)-a=0$.

I. Solution by W. E. HEAL, Marion, Ind.

Writing the equation $\frac{dy}{dx}+y\log x=a/x^x$, we have a linear equation of the standard form, and by the well known formula, $y=-Ca/x^x$.

II. Solution by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio.

Dividing by x^x and transposing we get $dy/dx+\log x.y=ax^{-x}$, the regular form of the *linear* equation.

Using Bernoulli's method, we proceed as follows:

Let $y=uz$; then $dy/dx=u\frac{dz}{dx}+z\frac{du}{dx}$, and the given equation becomes

$$u\frac{dz}{dx}+z(\frac{du}{dx}+u\log x)=ax^{-x}.$$

Now, if the function u be assumed of such value as to make the factor z in this equation vanish, we shall have the two equations,

$$\frac{du}{dx}+u\log x=0, \text{ and } u\frac{dz}{dx}=ax^{-x}.$$

From the former, we get by integration, $u=e^x x^{-z}$, and the value of u reduces the latter to $e^x\frac{dz}{dx}=a$ or $dz=ae^{-x}dx$. Whence $z=-ae^{-x}+C$.

Therefore $y=uz=e^x x^{-x}(-ae^{-x}+C)=x^{-x}(Ce^x-a)$, which is the required solution.

Also solved by *L. C WALKER, G. B. M. ZERR, F. P. MATZ*, and *J. SCHEFFER.*

131. Proposed by F P. MATZ. Sc. D.. Ph. D.. Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Integrate $2/x$, with regard to $d[\sqrt{(1-x^2)}]$.

Solution by the PROPOSER.

According to the conditions of the problem, we may write $d[F(x)]/d[\sqrt{(1-x^2)}]=2/x....(1)$.

$$\therefore F(x)=2\int\frac{d[\sqrt{(1-x^2)}]}{x}=-2\int\frac{dx}{\sqrt{(1-x^2)}}=\sec^{-1}\left[\frac{1}{2x^2-1}\right]$$

$=\cos^{-1}(2x^2-1)....(2)$, which is the integral required.

Also solved with various results by *H. C. WHITAKER, J. SCHEFFER*, and *G. B. M. ZERR.*

132. Proposed by JOHN M. COLAW, A. M., Monterey. Va.

What expression derived from the *polar* equation of a curve is equivalent to the expression for dy/dx derived from the *Cartesian* equation of the same curve? Prove work with $\rho=2r\cos\theta$.

Solution by LON C. WALKER. A.M., Professor of Mathematics, Petaluma High School. Petaluma, Cal.

Let $x=\rho\cos\theta$, $y=\rho\sin\theta$, then

$$\frac{dy}{dx}=\frac{\sin\theta(d\rho/d\theta)+\rho\cos\theta}{\cos\theta(d\rho/d\theta)-\rho\sin\theta}....(1).$$

From $\rho=2r\cos\theta$, we get $d\rho/d\theta=-2r\sin\theta$. Substituting this value of $d\rho/d\theta$ in (1), and reducing, we have

$$\frac{dy}{dx}=\frac{r-x}{y}....(2).$$

Now substituting the value of x and y in $\rho=2r\cos\theta$, we get $x^2+y^2=2rx$; from which $dy/dx=(r-x)/y$, the same as in (2).

The angle the tangent makes with the radius vector is $\rho(d\theta/d\rho)$, which is often used in the same way as dy/dx; for instance, in trajectories, etc.

Also solved by *G. B. M. ZERR*, and *J. SCHEFFER*.

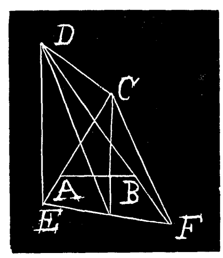
MECHANICS.

130. Proposed by W. J. GREENSTREET. M. A.. Editor of The Mathematical Gazette. Stroud. Gloucestershire, England.

Two particles are projected from A and B on the same level at α, β to horizon, and in vertical planes with which AB makes angles θ, φ. They meet and coalesce into a single particle. Find the height of the latus rectum of the snbsequent path above the level of A and B.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia. Pa.

Let D be the point of meeting of the two particles; mass of each, unity; C, the projection of D on the plane of AB; DE, DF, the tangents to the two paths, respectively; G, the mid-point of EF. Then DG is the direction of the resultant of DE and DF. Let $AB=a$, $\angle CAB=\theta$, $\angle CBA=\varphi$.



Then $AC=a\sin\theta/\sin(\theta+\varphi)$, $BC=a\sin\varphi/\sin(\theta+\varphi)$.

The equation to the path through AD is $m=n\tan\alpha-gn^2/2v^2\cos^2\alpha$, when $n=a\sin\theta/\sin(\theta+\varphi)$.

$$m=\frac{a\sin\theta\tan\alpha}{\sin(\theta+\varphi)}-\frac{ga^2\sin^2\theta}{2v^2\sin^2(\theta+\varphi)\cos^2\alpha}=h\ldots.(1).$$

The equation to the path through BD is $p=q\tan\beta-gq^2/2v_1^2\cos^2\beta$, when $q=a\sin\varphi/\sin(\theta+\varphi)$;

$$p=\frac{a\sin\varphi\tan\beta}{\sin(\theta+\varphi)}-\frac{ga^2\sin^2\varphi}{2v_1^2\sin^2(\theta+\varphi)\cos^2\beta}=h\ldots.(2).$$

From (1) and (2),

$$v_1^2=\frac{agv^2\sin^2\varphi\cos^2\alpha}{\cos^2\beta[ga\sin^2\varphi+2v^2\cos^2\alpha\sin(\theta+\varphi)(\sin\varphi\tan\beta-\sin\theta\tan\alpha)]}.$$

$\tan DEC=dm/dn=\tan\alpha-gn/v^2\cos^2\alpha=\tan A$.

$$\therefore \tan A=\tan\alpha-\frac{ag\sin\theta}{v^2\sin(\theta+\varphi)\cos^2\alpha};\ \tan DCF=\frac{dp}{dq}=\frac{\tan\beta-gq}{v_1^2\cos^2\beta}=\tan B.$$

$$\therefore \tan B=\tan\beta-\frac{ag\sin\varphi}{v_1^2\sin(\theta+\varphi)\cos^2\beta}.$$

Let velocity at D for particle from $A=u$, for particle from $B=u_1$; then $u^2=v^2-2gh$, $u_1^2=v_1^2-2gh$. Now $CE=h\cot A$, $CF=h\cot B$.

$\therefore$ $EF=h\sqrt{[\cot^2 A+\cot^2 B+2\cot A\cot B\cos(\theta+\varphi)]}=d$.

$CG=\frac{1}{2}\sqrt{[2CE^2+2CF^2-EF^2]}=\frac{1}{2}h\sqrt{[\cot^2 A+\cot^2 B-2\cot A\cot B\cos(\theta+\varphi)]}=l$.

$\therefore$ $\tan DGC=h/l=\tan C$, where C=angle of projection of coalesced particles at D. Also $DE=h\operatorname{cosec} A$, $DF=h\operatorname{cosec} B$.

$$\therefore\ \cos EDF=\frac{h^2\operatorname{cosec}^2 A+h^2\operatorname{cosec}^2 B-d^2}{2h^2\operatorname{cosec} A\operatorname{cosec} B}=\cos D.$$

$\therefore$ $\cos D=\sin A\sin B-\cos A\cos B\cos(\theta+\varphi)$.

Let w=the velocity of the two particles after they coalesce.

Then $w=\sqrt{[u^2+u_1^2+2uu_1\cos D]}$.

$\therefore$ Their path is $y=x\tan C-gx^2/2w^2\cos^2 C$.

Ordinate of vertex$=w^2\sin^2 C/2g$.

Latus rectum$=-2w^2\cos^2 C/g$.

Height of latus rectum above $D=\dfrac{w^2}{2g}(\sin^2 C-\cos^2 C)=\dfrac{w^2}{2g}(1-2\cos^2 C)=k$

Height of latus rectum above plane of $AB=h+k$.

131. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

If the distributed weight on the foundations of a building is W lb./(feet)^2, the foundations must be sunk $D=(W/w)\tan^4(\frac{1}{4}\pi-\frac{1}{2}\phi)$ feet deep in earth of density $w$lb./(feet)^3 and angle of repose ϕ.

No solution of this problem has been received.

132. Proposed by T. U. TAYLOR, C. E., Department of Engineering, University of Texas, Austin, Tex.

1. A parabola, whose axis is vertical, is described on the vertical face of a reservoir wall. If the vertex O of the parabola is at the bottom of the wall, and the parabola intersects the surface in the points A, B, find the depth of the center of pressure of the water on the parabolic area ABO.

2. In the same problem find the center of pressure on the area included between the horizontal line through O, a vertical through B, and the curve OB.

Solution by P. H. PHILBRICK, C. E., Lake Charles, La.

The general formula for the center of pressure, is

$$\bar{x}=\frac{\int\int hx\,dxdy}{\int\int h dxdy}$$

in which h is the depth of any point below the surface.

Let O be the origin, its depth being a, and the ordinate on the surface $=b$.

Let $y^2=m^2x$ be the equation of the curve. In this case we have,

$$\bar{x}=\frac{\int_0^a\int_0^{mx^{\frac{1}{2}}}(a-x)xdxdy}{\int_0^a\int_0^{mx^{\frac{1}{2}}}(a-x)dxdy}=\frac{\int_0^a(a-x)x^{\frac{3}{2}}\,dx}{\int_0^a(a-x)x^{\frac{1}{2}}\,dx}=\left[\frac{\frac{2}{5}ax^{\frac{5}{2}}-\frac{2}{7}x^{\frac{7}{2}}}{\frac{2}{3}ax^{\frac{3}{2}}-\frac{2}{5}ax^{\frac{5}{2}}}\right]_0^a=\tfrac{3}{7}a.$$

Therefore, the center of pressure is $\frac{4}{7}a$ below the surface.

(2). To find the center of pressure on the outer figure. We have

$$\bar{x}=\frac{\int_0^a\int_0^{b-mx^{\frac{1}{2}}}(a-x)xdxdy}{\int_0^a\int_0^{b-mx^{\frac{1}{2}}}(a-x)dxdy}=\frac{\int_0^a(a-x)(b-mx^{\frac{1}{2}})xdx}{\int_0^a(a-x)(b-mx^{\frac{1}{2}})dx}$$

$$=\left[\frac{\frac{1}{2}abx^2-\frac{2}{5}amx^{\frac{5}{2}}-\frac{1}{3}bx^3+\frac{2}{7}mx^{\frac{7}{2}}}{abx-\frac{2}{3}amx^{\frac{3}{2}}-\frac{1}{2}bx^2+\frac{2}{5}mx^{\frac{5}{2}}}\right]_0^a.$$

But the curve gives $b^2=m^2a$ or $m=b/a^{\frac{1}{2}}$. Substituting this value of m, and reducing, we have, $\bar{x}=\frac{11}{49}a$. Hence the depth of the center of pressure$=\frac{38}{49}a$.

Also solved by *G. B. M. ZERR*, and *T. T. DAVIS*.

DIOPHANTINE ANALYSIS.

89. Proposed by J. H. DRUMMOND. LL. D., Portland, Me.

Show that in $2x^2+2y^2-z^2=\square\ldots.(1)$,

$2x^2+2z^2-y^2=\square\ldots.(2)$,

$2y^2+2z^2-x^2=\square\ldots.(3)$,

any two numbers and their sum and difference will satisfy the conditions.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia. Pa.

Let $2x^2+2y^2-z^2=a^2$, $2x^2+2z^2-y^2=b^2$, $2y^2+2z^2-x^2=c^2$.

Adding, $2x^2+2y^2+2z^2=\frac{2}{3}(a^2+b^2+c^2)$.

$\therefore z=\pm\frac{1}{3}\sqrt{(2b^2+2c^2-a^2)}$,

$y=\pm\frac{1}{3}\sqrt{(2a^2+2c^2-b^2)}$,

$x=\pm\frac{1}{3}\sqrt{(2a^2+2b^2-c^2)}$.

Let $b=(n+1)a$, $c=na$.

$\therefore z=\pm\frac{a}{3}(2n+1)$, $y=\pm\frac{a}{3}(n-1)$, $x=\pm\frac{a}{3}(n+2)$.

Solutions of problem 87 were received from *H. S. VANDIVER*, *G. B. M. ZERR*, *J. H. DRUMMOND*, and *H. C. WHITAKER*; of problem 88. from *G. B. M. ZERR*, *J. H. DRUMMOND*, *J. SCHEFFER*, and *H. C. WHITAKER*.

90. Proposed by HARRY S. VANDIVER. Bala, Pa.

Prove that it is always possible to find an infinite number of positive integral values of x, y, and z, such that the relation $z^2=x^2+bxy+cy^2$ is satisfied, b and c being any integers whatever.

Solution by the PROPOSER.

If $x^2+bxy+cy^2=z^2$....(1), then solving for x,

$$x=\frac{-b\pm\sqrt{[b^2y^2-4(cy^2-z^2)]}}{2}.$$

If x is to be a positive integer it follows that $b^2y^2-4cy^2+4z^2=$a square. So that the solution of the original problem is reduced to a solution of

$$X^2+AY^2=Z^2....(2).$$

It may be shown directly that this relation possesses an infinite number of integral solutions for X, Y, and Z, $-A$ being any integer whatever.

$$(m^2+An^2)^k=X^2+AY^2$$
$$\text{for } (m+n\sqrt{-A})^k=X+Y\sqrt{-A}$$
$$\text{and } (m-n\sqrt{-A})^k=X-Y\sqrt{-A}$$

whence. by multiplication, $m^2+An^2=X^2+AY^2$.

Hence we may assume in (2) that $Z=m^2+An^2$; m and n being any numbers. Then

$$(m^2+An^2)^2=X^2+AY^2$$
$$\text{and } (m+n\sqrt{-A})^2=X+Y\sqrt{-A}.$$

Expanding, we have $m^2-n^2A+2mn\sqrt{-A}=X+Y\sqrt{-A}$. Whence

$$X=m^2-n^2A$$
$$Y=2mn$$
$$\text{and by assumption, } Z=m^2+An^2.$$

Hence, (2) has an infinite number of solutions, since a solution is found for every value given to m and n. Also, for every solution of (2) there is a solution of (1), hence (1) has an infinite number of solutions, which was to be proved.

Proceeding in like manner, it can be shown that the relation

$$x^2+bxy+cy^2=z^n....(3) \ (n, \text{ an integer})$$

also possesses an indefinite number of solutions, but the method which has been used does not enable us to find *all* the solutions of either (1) or (3).

91. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

There are two unequal square numbers the sum of whose sum, difference, product, and quotient, is a square. Find the two numbers.

Solution by J. H. DRUMMOND, LL. D., Portland, Me.

Any two unequal squares answer the conditions of the question; for, let a^2 and b^2 be the numbers, then $a^2+b^2+a^2-b^2+a^2b^2+(a^2/b^2)$ must be a square. Reducing, and dropping the factor a^2, we have $2+b^2+(1/b^2)$, a square. This is readily put under the form $\dfrac{b^4+2b^2+1}{b^2}$, which is the square of $\dfrac{b^2+1}{b^2}$.

Q. E. D.

Also solved in a similar manner by *G. B. M. ZERR*, *J. SCHEFFER*, and *H. S. VANDIVER*.

92. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

(*a*) Find the least three integral numbers such that the difference of every two of them shall be a square number; (*b*) Find the least three square numbers such that the difference of every two of them shall be a square number.

Solution by J. H. DRUMMOND, LL. D., Portland, Me.

Part 2. Let x^2, m^2x^2, and n^2x^2 be the three numbers; then m^2-1, n^2-1, and m^2-n^2 must all be squares.

Assume $m=\dfrac{p^2+1}{p^2-1}$ and $n=\dfrac{q^2+1}{q^2-1}$, and m^2-1 and n^2-1 will be squares and it remains to make $\left[\dfrac{p^2+1}{p^2-1}\right]^2-\left[\dfrac{q^2+1}{q^2-1}\right]=\square$, or reducing $(p^2q^2-1)(\dfrac{q^2}{p^2}-1)=\square$.

This is done by making $pq=\dfrac{r^2+s^2}{2rs}$ and $\dfrac{q}{p}=\dfrac{t^2+u^2}{2tu}$.

But $pq\times q/p=q^2=\left[\dfrac{r^2+s^2}{2rs}\right]\left[\dfrac{t^2+u^2}{2tu}\right]$. Hence $rs(r^2+s^2)tu(t^2+u^2)$ must be a square. Take $r=f+g$, $s=f-g$, $t=h+k$, and $u=h-k$, and substituting and reducing, we have $(f^4-g^4)(h^4-k^4)=\square$. Assume $f^2=h^4-k^4+v^2$ and $g^2=h^4-k^4-v^2$ and $(f^4-g^4)=4v^2(h^4-k^4)$, and $(f^4-g^4)(h^4-k^4)$ becomes $4v^2(h^4-k^4)^2$, a square. Assume $h^2=a^2v^2$ and $k^4=b^2v^2$; then $h^2=av$ and $k^2=bv$. But $f^2-g^2=2v^2$. Assume $f+g=4v$ and $f-g=\frac{1}{2}v$, then $f=\frac{9}{4}v$ and $g=\frac{7}{4}v$.

Then $r=4v$ and $s=\frac{1}{2}v$. Then $pq\left[=\dfrac{r^2+s^2}{2rs}\right]=\frac{65}{16}$, and $q/p=\dfrac{t^2+u^2}{2tu}$ $=\dfrac{2(h^2+k^2)}{2(h^2-k^2)}=\dfrac{a+b}{a-b}$; and $h^4-k^4+v^2=f^2=\dfrac{81v^2}{16}$, and $h^4-k^4=\dfrac{65v^2}{16}$. Therefore $a^2-b^2=\dfrac{h^4-k^4}{v^2}=\frac{65}{16}$. Now $pq\times q/p=q^2=\dfrac{65(a+b)}{16(a-b)}=\dfrac{65(a+b)^2}{16(a-b)^2}=(a+b)^2$ and $q=(a+b)$ and $p=(a-b)$. Hence $m=\dfrac{(a-b)^2+1}{(a-b)^2-1}$ and $n=\dfrac{(a+b)^2+1}{(a+b)^2-1}$, in which a and b may be any square numbers which make $a^2-b^2=\frac{65}{16}$.

It would seem that $a^2-b^2=\frac{65}{16}$ ought to lead readily to a general solution, but a and b were both so taken that they must be squares; b is readily found to

be $\dfrac{13-5c^2}{8c}$. Hence $26c-10c^3$ must be a square; it is evident that this is the case when $c=1$. Then $b=1$ and $a=\frac{9}{4}$. Substituting these in the values of m and n, and we have $m=\frac{41}{9}$ and $n=\frac{185}{153}$. Taking $x=153$, we have $mx=697$, and $nx=185$, and the numbers are $(153)^2$, $(185)^2$, and $(697)^2$.

Also solved by *G. B. M. ZERR*, and *EDWARD D. GRABER*.

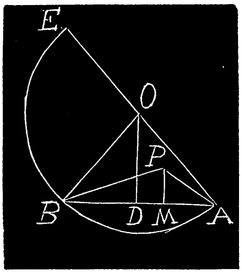
AVERAGE AND PROBABILITY.

111. Proposed by LON C. WALKER, A.M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

If a radius be drawn at random in a given semi-circle, and a point taken at random in one of the sectors formed, show that the chance that a random line drawn through the point will cut the arc of the sector is $1-\frac{1}{\pi^2}\log 2$.

Solution by the PROPOSER.

Let ABE be the given semicircle, OB the random radius, P the random point, OD and PM perpendicular to AB.



Put $DM=x$, $PM=y$, $OA=1$, $\angle AOB=\theta$, $\angle APM=\phi$, $\angle BPM=\psi$. Then $AD=\sin\frac{1}{2}\theta$, $OD=\cos\frac{1}{2}\theta$, area of segment $ACB=\frac{1}{2}(\theta-\sin\theta)$, $\phi=\tan^{-1}\left(\dfrac{\sin\frac{1}{2}\theta-x}{y}\right)$, $\psi=\tan^{-1}\left(\dfrac{\sin\frac{1}{2}\theta+x}{y}\right)$

When P is in the segment ACB the random line will cut the arc whatever be its direction, and when P is in the triangle AOB the number of favorable directions of the random line will be $2(\phi+\psi)$. Hence we have

$$p=\frac{\displaystyle\int_0^\pi \pi(\theta-\sin\theta)d\theta+\int_0^\pi\int_0^{\cos\frac{1}{2}\theta}\int_{-\tan\frac{1}{2}\theta(\cos\frac{1}{2}\theta-y)}^{\tan\frac{1}{2}\theta(\cos\frac{1}{2}\theta-y)}2(\phi+\psi)dxdyd\theta}{\displaystyle\int_0^\pi \pi\theta d\theta}$$

$$=1-\frac{4}{\pi^2}+\frac{4}{\pi^3}\int_0^\pi\int_0^{\cos\frac{1}{2}\theta}\Bigg[2\tan\tfrac{1}{2}\theta(2\cos\tfrac{1}{2}\theta-y)\tan^{-1}\tan\tfrac{1}{2}\theta\left(\frac{2\cos\frac{1}{2}\theta-y}{y}\right)$$

$$-y\theta\tan\tfrac{1}{2}\theta-y\log\left(\frac{y^2+\tan^2\frac{1}{2}\theta(2\cos\frac{1}{2}\theta-y)^2}{2y^2}\right)\Bigg]dy\,d\theta$$

$$=1-\frac{4}{\pi^2}+\frac{4}{\pi^3}\int_0^\pi\Bigg[\theta\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta-2(\theta-\pi)\sin^3\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta-2\sin^2\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\theta\log^2$$

$$+\tfrac{1}{4}\log\left(\frac{1+\cos\theta}{1-\cos\theta}\right)+\tfrac{1}{4}\cos\theta\log(1+\cos\theta)+\tfrac{1}{2}\cos2\theta\log(1-\cos\theta)\Bigg]d\theta=1-\frac{1}{\pi^2}\log 2.$$

Solved with same result by *F. P. MATZ*. Professor Zerr gets as a result $1-(1/4\pi^2)(8\log2+7)$.

112. Proposed by LON C. WALKER, A.M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Two circles are drawn at random, both in magnitude and position, but so as to lie wholly upon the surface of a given circle. Show that the chance of their both resting on the same diameter of the given circle is $\frac{4}{\pi}(\log 2-5)$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let A be the center of one of the circles, $AO=x$, $AB=y$, z=radius of other circle. With center O and radius $OK=r-z$ describe a circle. Draw MQ, NP through O tangent to circle center A.

Also draw CE, DH, CF, DG parallel to NP, MQ, respectively, at a distance z from them. If center of the second circle lies on either $CEKFC$ or $GDHLG$, both circles will not lie on the same diameter.

Area $CEKFC$+area $GDHLG$=

$$2\left[(r-z)^2\left[\cos^{-1}(y/x)-\sin^{-1}\left(\frac{z}{r-z}\right)\right]-z\surd(r^2-2rz)+\frac{yz^2}{\surd(x^2-y^2)}\right]=u.$$

The limits of z are 0 and $r\surd(x^2-y^2)/[x+\surd(x^2-y^2)]=z'$; of y, 0 and x, and 0 and $(r-x)$; of x, 0 and $\frac{1}{2}r$, and $\frac{1}{2}r$ and r.

$$\therefore p=1-\frac{\left[\int_{\frac{1}{2}r}^{r}\int_0^{r-x}2\pi xdxdy+\int_0^{\frac{1}{2}r}\int_0^{x}2\pi xdxdy\right]\int_0^{z'}udz}{\int_0^r\int_0^{r-x}\int_0^r\pi(r-z)^2.2\pi xdxdydz}$$

$$=1-\frac{18}{\pi r^6}\left[\int_{\frac{1}{2}r}^{r}\int_0^{r-x}xdxdy+\int_0^{\frac{1}{2}r}\int_0^{x}xdxdy\right]\int_0^{z'}udz$$

$$=1-\frac{4}{5\pi r^3}\left[\int_{\frac{1}{2}r}^{r}\int_0^{r-x}\{15\cos^{-1}(y/x)-10+\frac{5}{y^5}[3x^2+3x\surd(x^2-y^2)-y^2]\right.$$

$$\times[x-\surd(x^2-y^2)]^3\}xdxdy+\int_0^{\frac{1}{2}r}\int_0^{x}\{15\cos^{-1}(y/x)-10$$

$$\left.+\frac{5}{y^5}[3x^2+3x\surd(x^2-y^2)-y^2][x-\surd(x^2-y^2)]^3\}xdxdy\right].$$

Let $y=x\sin\theta$.

Then $\theta=0$ and $\sin^{-1}\left[\frac{r-x}{x}\right]=\theta'$.

$$\therefore p=1-\frac{4}{5\pi r^3}\left[\int_{\frac{1}{2}r}^{r}\int_0^{\theta}[15(\tfrac{1}{2}\pi-\theta)\cos\theta-10\cos\theta+10\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta-\tfrac{5}{2}\tan\tfrac{1}{2}\theta\sec^2\tfrac{1}{2}\theta]\times\right.$$

$$\left.x^2dxd\theta+\int_0^{\frac{1}{2}r}\int_0^{\frac{1}{2}\pi}[15(\tfrac{1}{2}\pi-\theta)\cos\theta-10\cos\theta+10\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta-\tfrac{5}{2}\tan\tfrac{1}{2}\theta\sec^2\tfrac{1}{2}\theta]x^2dxd\theta\right]$$

$$=1-\frac{4}{5\pi r^3}\left[\int_{\frac{1}{2}r}^{r}\left(15(r-x)\cos^{-1}(\frac{r-x}{x})-10r-20\sqrt{(2rx-r^2)}\right.\right.$$

$$\left.\left.+\tfrac{65}{2}x-\frac{5x^2}{x+\sqrt{(2rx-r^2)}}\right)xdx+\tfrac{15}{2}\int_0^{\frac{1}{2}r}x^2dx\right]=\frac{4}{\pi}(8\log2-5).$$

Also solved with same result by the *PROPOSER*.

MISCELLANEOUS.

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107. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

The index of refraction of a medium varying inversely as the square root of the distance, prove that the path of a ray of light in the medium is a cycloid.

Solution by the PROPOSER.

Taking the axis of y in the given plane and that of x at right angles to the y axis, letting $\mu=k/\sqrt{x}$ be the index of refraction, and $p=dy/dx$, we have, by the usual theory, for the differential equation to the path

$$\frac{dp/dx}{1+p^2}=\frac{1}{\mu}\left[\frac{d\mu}{dy}-\frac{d\mu}{dx}\frac{dy}{dx}\right]\ldots.(1). \quad \text{We have } \frac{d\mu}{dx}=-\frac{k}{2x^{\frac{3}{2}}}\ldots.(2),$$

and (1) becomes $\dfrac{dp/dx}{1+p^2}=\dfrac{p}{2x}$, or $\dfrac{dp}{p(1+p^2)}=\dfrac{dx}{2x}\ldots.(3)$.

Integrating, $\log\dfrac{p}{\sqrt{(1+p^2)}}=\log\sqrt{x}+C\ldots.(4)$.

Let $p=b$, when $x=a$; then $C=\log\dfrac{b}{a\sqrt{(1+b^2)}}$,

and (4) becomes $p=\dfrac{dy}{dx}=\dfrac{xdx}{\sqrt{[(a^2/b^2)(1+b^2)x-x^2]}}\ldots.(5)$, the differential equation to a cycloid.

Also solved by *G. B. M. ZERR*, and *L. C. WALKER*.

108. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

To divide the arc of a cardioid into eight equal parts.

Solution by L. C. WALKER, A.M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

Equation of the cordioid is $r=a(1+\cos\theta)$, and its complete perimeter is

$$S=2\int_0^\pi \sqrt{[a^2(1+\cos\theta)^2+a^2\sin^2\theta]}d\theta=4a\int_0^\pi \cos\tfrac{1}{2}\theta d\theta=8a,$$

the arc S being measured from the point where the curve crosses the initial line.

Let θ' be the point of the first division; then

$$2a\int_0^{\theta'}\cos\tfrac{1}{2}\theta d\theta=4a\sin\tfrac{1}{2}\theta'=a;$$

from which $\theta_{1,\,7}=\pm2\sin^{-1}\frac{1}{4}$. Similarly, $\theta_{2,\,6}=\pm2\sin^{-1}\frac{1}{2}$, $\theta_{3,\,5}=\pm2\sin^{-1}\frac{3}{4}$, $\theta_4=\pi$, and $\theta_8=0$.

Excellent solutions were received from *G. B. M. ZERR*, and *F. P. MATZ*.

109. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the latitude of the place where the sun's center remains above the horizon for a hundred successive days.

Solution by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy, Defiance College, Defiance, O.

I. The solar phenomenon in question will begin May 2 and end August 10. For May 2 at Washington mean noon, according to the Director of United States Nautical Almanac Office, the declination of the sun is $\delta=+15°\ 18'\ 54.7''$. Representing the hour-angle of the sun by h, we have $\cos h=-\tan\lambda\tan\delta$....(1).

Since there is to be no setting of the sun, we may put $\cos h=-1$.

$\therefore\ \lambda=\tan^{-1}\left(\dfrac{1}{\tan\delta}\right)=\tan^{-1}(\cot\delta)=90°-\delta,=74°\ 41'\ 5.3''$, which is the terrestrial latitude of the place required. This solar phenomenon may be observed on Melville Island.

II. The celestial longitude of the sun for mean noon Washington can be determined in various ways to be about $\psi=41°\ 34'\ 25''$; the obliquity of the ecliptic may be taken $\omega=23°\ 27'\ 8''$; then $\lambda=90-\delta=90-\sin^{-1}(\sin\psi\times\sin\omega),=74°\ 43'\ 16''$.

Also solved by *G. B. M. ZERR*, and *H. C. WHITAKER*.

110. Proposed by E. W. MORRELL, South Trowbridge, Vt.

If a and b be the sides of a triangle, A and B the angles opposite, then will $\log b-\log a=\cos3A-\cos2B+\frac{1}{2}(\cos4A-\cos4B)+\frac{1}{3}(\cos6A-\cos6B)+....$

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

$1+\cos2A+\frac{1}{2}\cos4A+\frac{1}{3}\cos6A+....=-\frac{1}{2}\log(2-2\cos2A)=-\log(2\sin A)$.

$1+\cos2B+\frac{1}{2}\cos4B+\frac{1}{3}\cos6B+....=-\log(2\sin B)$.

See Trigonometry for the summation of these series.

$\therefore\ \cos2A-\cos2B+\frac{1}{2}(\cos4A-\cos4B)+....=\log(\sin B/\sin A)=\log(b/a)$.

Also solved by *F. P. MATZ*, and *J. SCHEFFER*.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

158. Proposed by JAMES F. LAWRENCE, A. B., Professor of Mathematics, Rogers Academy, Rogers, Ark.

My agent sold pork at 5% commission; increasing the proceeds by $20, I ordered the purchase of flour at 3% commission; after which flour rose 9%, my whole gain was $40. What did he sell the pork for?

159. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

The amount of tax assessed on the property of a city is T,=$145850; and the treasurer was allowed a fee of $m\%$,$=\frac{3}{4}\%$, for collection. If $n\%$,=10%, of the tax was uncollectible, what were the net proceeds of the tax?

ALGEBRA.

153. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

If $x=\Sigma_0^\infty e^{-k[t+(2a\pi/h)]}\sin n\left(t+\dfrac{2a\pi}{h}\right)$, find value of x freed from Σ_0^∞.

154. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Deduce the Sylvestrian Reciprocant from $ax^3+3bx^2y^2+ay^3+d=0$.

GEOMETRY.

187. Proposed by R. TUCKER, M. A.

AD, BE, CF are the altitudes of the triangle ABC; k_1, k_1'; k_2, k_2'; k_3, k_3' are the S points of the triangles EAB, FCA; FBC, DAB; DCA, EBC, respectively; prove that $k_3'k_1=k_1'k_2=k_2'k_3=R\sin A\sin B\sin C$. ρ_1, ρ_1'; ρ_2, ρ_2'; ρ_3, ρ_3' are the Brocard radii of the above triangles, prove that (1) $\rho_1\rho_2\rho_3=\rho_1'\rho_2'\rho_3'$; (2) $(\rho_2'^2-\rho_3^2)/a^2+(\rho_3'^2-\rho_1^2)/b^2+(\rho_1'^2-\rho_2^2)/c^2=\frac{3}{64}$; (3) the sets of 4 Brocard-points for the above pairs of triangles are concyclic (on three circles); (4) the tangent from any one of the right angles of the above triangles to the Brocard circle of the triangle is a mean proportional between the tangents to the same circle from the remaining (two) angles.

188. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

$ABCD$ is a quadrilateral whose diagonal triangle is PQR, P on AD and R on AB. PQ meets AB in Z. If C moves along PB what will happen to Z?

CALCULUS.

152. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy, Defiance College, Defiance, Ohio.

Solve the linear differential equation

$$e^x\left[\frac{dy}{dx}-y\log x\right]-a[\log x+1]=0.$$

153. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the equation of the loxodromic curve on an oblate spheroid.

MECHANICS.

142. Proposed by GEORGE R. DEAN, B. Sc., Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

In infinite mass of liquid is bounded by the plane zx, on which are small corrugations given by $y=\phi(x)$. The velocity of the liquid at an infinite distance from the plane is parallel to x and equal to V. Prove that the velocity potential is $V_x+\dfrac{V}{\pi}\displaystyle\int_{-\infty}^{\infty}\frac{(x-\lambda)\phi(\lambda)d\lambda}{y^2+(x-\lambda)^2}$. [Bassett's *Hydrodynamics.*]

143. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

Beads are fastened at equal intervals on a string placed over a smooth fixed pulley. If the original position of the string is one of symmetry, find the velocity at any moment, the pressure on the pulley, and the velocity with which the string leaves the pulley.

DIOPHANTINE ANALYSIS.

102. Proposed by F. L. SAWYER, Mitchel, Ontario, Canada.

Prove that the factors of the sum of the squares of two numbers prime to each other are themselves the sum of two squares.

103. Proposed by HARRY S. VANDIVER, Bala, Pa.

Find some solutions of $x^3+ay^3=z^2$ (for x, y, and z) and show that there is an infinite number of solutions corresponding to each integral value of a.

AVERAGE AND PROBABILITY.

127. Proposed by G. B M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

What is the probable error of the volume of a rectangular parallelopiped whose edges measured by the repeated application of a unit of measure are found to be a, b, c, supposing that the probable error of a line so measured whose length is found to be l is $r\sqrt{l}$.

128. Proposed by G. B. M. ZERR, A. M., Ph. D., Profeseor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Two small circles are drawn on the surface of a sphere so as to intersect; find average area of the spherical triangle formed by joining the poles and one of the intersections of the small circles with arcs of great circles.

BOOKS AND PERIODICALS.

Four-Place Logarithmic Tables containing the Logarithms of Numbers and of the Trigonometric Functions. Arranged for use in the entrance examinations of the Sheffield Scientific School of Yale University. By Percy F. Smith. New York: Henry Holt & Co.

Elementary Calculus. A text-book for the use of students in General Science. By Percy F. Smith, Ph. D., Professor of Mathematics in the Sheffield Scientific School of Yale University. 8vo. Cloth, 89 pages. Price, 80 cents. New York and Chicago: The American Book Co.

The aim of this book is to satisfy the growing demand for a text-book on the Calculus which shall present in a course of from thirty-five to forty exercises the fundamental notions of the Calculus. The remarkable development of Physics and Chemistry, not to mention other important subjects, along mathematical lines, makes a working knowledge of the Calculus an absolute necessity. This book can easily be mastered in the time assigned by any one having a slight aptitude for mathematics.

Physics for High School Students. By Henry S. Carhart, LL. D., Professor of Physics in the University of Michigan, and Horatio N. Chute, M. S., Instructor of Physics in the Ann Arbor High School. 8vo. Cloth, vii+433 pages. Price, $1.25. Boston: Allyn & Bacon.

This volume, the authors say, was written with the same purpose that guided them in the preparation of their *Elements of Physics.* The Elements was a most excellent book, well adapted to the wants of the schools for which it was written. In bringing out this book, the rapid advances in Physics within the last five or six years made it necessary to rewrite the whole book, and thus make it an entirely new one. The authors have made great improvements, and those teachers who, like myself, found the Elements a most excellent work for academic purposes, will find this one even better and more helpful.

Annals of Mathematics. Founded by Ormond Stone. Published under the auspices of Harvard University. Second Series, Vol. 3, No. 3.

The April number contains: Space of Constant Curvature, by Prof. F. S. Wood; Brilliant Points and Loci of Brilliant Points, by W. H. Roever; Problems in Infinite Series and Definite Integrals, by Prof. W. F. Osgood; Note on the Product of Linear Substitutions, by Prof. H. B. Newson.

The American Journal of Mathematics. Edited by Frank Morley with the co-operation of Simon Newcomb and other mathematicians. Published under the auspices of Johns Hopkins University. Price, $5.00.

Number 2, Vol. XXIV, contains Canonical Form of Linear Homogeneous Transformation in an Arbitrary Realm of Rationality, by Dr. L. E. Dickson; A New Theory of Collineation and their Lie Groups, by H. B. Newson; Infinitesimal Deformation of Surfaces, by L. P. Eisenhart.

ERRATA.

Page 66, line 7, from top of page, for "oreer" read *order*.
Page 67, line 15, from top of page, for "$\alpha-\beta+\delta-\varepsilon$" read $\alpha-\beta+\gamma-\varepsilon$.
Page 70, line 1, for "$x=22$" read $x=17$.

THE

AMERICAN

MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. IX. MAY, 1902. No. 5.

MODELS OF THE WEIERSTRASS SIGMA FUNCTION AND THE ELLIPTIC INTEGRAL OF THE SECOND KIND.

By DR. VIRGIL SNYDER, Cornell University.

In 1886, Professor Dyck, of the Polytechnicum of Munich, requested the construction of models to represent the elliptic functions. In accordance with this request H. Burkhardt and M. W. Wildbrett constructed seven models, two of which are for the case in which the invariant J of the binary quartic is equal to zero.

In 1899, Professor Klein, while lecturing on the automorphic functions, requested that the series be extended by making corresponding models of the sigma function, and of the elliptic integral of the second kind.

By Professor Klein's invitation I undertook the task which consists of three steps. These may be named as follows:

(1) Analytic representation;

(2) Numerical calculation;

(3) Mechanical construction.

(1) ANALYTIC REPRESENTATION. The sigma function can most easily be expressed in terms of theta functions, which in turn are defined by trigonometric series with exponential functions for coefficients. For values of the variable within the parallelogram containing the origin two terms of the series are sufficient for numerical values correct to three places of decimals. For values of the

variable not in the initial parallelogram more terms are necessary, but the form of the function can be easily expressed by a recurring formula. Since three terms are sufficient to obtain approximations correct to eight places of decimals in the first parallelogram, it was found that three terms are sufficient for all values included in nine parallelgrams about the origin.

The two components of the variable are chosen as rectangular coördinates in a plane, and the real part of the function is erected as ordinate from the point.

Thus, if $u+iv=f(x+iy)$, then $u=f_1(x, y)$, and $v=f_2(x, y)$.

The following theorems can now be easily established when the invariant J is put equal to zero.

The line $x=0$, $u=0$ lies on the surface $u=f_1(x, y)$.

The u-surface is symmetric as to the plane $x=0$.

Finally, $u(-x, y)=-u(x, y)$.

To obtain the v surface, we have the relation

$$f(iz)=if(z),$$

hence the v surface can be obtained from the u surface by rotating the latter through a positive right angle about the line $x=0$, $y=0$.

(2) Numerical Calculation. By means of logarithmic and trigonometric tables and also tables for exponential and hyperbolic functions the numerical calculation is a matter of routine. I gave y a constant value, then calculated the value of u for values of x taken at intervals of $\frac{1}{24}$ of the period; this was repeated for values of y taken at intervals of $\frac{1}{24}$ of the second period. The numerical values were thus determined over nine parallelograms. The niveau lines are now easily found by marking those points on the parallelogram which have the same value of u. By drawing the curves which connect points of the same value of u the entire system is determined. To one familiar with topographical drawings this system of niveau curves furnishes a very good representation of the function.

(3) Mechanical Construction. When $J=0$ the two periods are numercally equal and their ratio is i, hence they may be represented by the sides of a square. For length of period 6 c.m. was chosen so that the base is 18 x 18 c.m. The curves $y=c$, $x=k$ were drawn on cross-section paper ruled to millemeters. The squares of paper were pasted on sheets of zinc and the zinc was then sawed through along the curve. Only alternate plane sections were drawn, at intervals of 5 m.m. These curves in zinc were then dove-tailed together, carefully squared and firmly soldered. A base was then sawed out corresponding to $z=-9$ (c.m.) and fixed to the previous part. This frame was filled with plaster, which was then shaved off while still green until the curves $y=c$, $x=k$ were exposed. After the plaster became more firmly set the exact contour was given to the surface by checking off the ordinates of the intermediate points. The niveau lines were easily marked by adjusting a horizontal scratch-awl so as to make a slight furrow in the plaster at a constant distance above the base. In this way

niveau lines at intervals of 1 c.m. were designated. A rather more elaborate mechanism is necessary to mark the lines of flow v=constant. These lines are so drawn that their vertical projections on the x, y plane are orthogonal to the niveau curves u=constant.

The model, thus constructed, was sent to Hrn. Kreittmäyr in Munich, for duplication. During the summer of 1899 I worked for several days in his workshops but didn't learn enough to make a complete report of this interestiug process.

The elliptic integral of the second kind is defined as the logarithmic derivative of the sigma function. The procedure for this function was essentially the same as for the sigma function, but here four parallelograms of periods are sufficient, as the function is periodic with regard to one of the periods of the elliptic function. The three theorems for sigma still hold for this function.

One fact is at once apparent from the model of the sigma function; for values of z within the first parallelogram it is of practical value in numerical calculations, but comparatively cumbersome for the next parallelogram and utterly worthless for parallelograms still further removed.

Both models are published by Martin Schilling, in Halle, successor to S. Brill, in Darmstadt.

MERIDIAN AND TRANSVERSE SECTIONS OF HELICOIDS OF UNIFORM PITCH.

By ALFRED HUME, C. E.. D. Sc., Professor of Mathematics, University of Mississippi, University, Miss.

The following discussion was suggested by the treatment of this subject given in MacCord's Descriptive Geometry. In addition to establishing analytically the facts stated in that text-book, other properties of the curves of intersection have been determined.

The helicoid here considered is a surface generated by a straight line always tangent to a right circular cylinder, inclined at a constant angle to the axis of the cylinder, and having two simultaneous uniform motions—one of revolution about the axis, the other of translation parallel thereto. The locus of the point of tangency of this generatrix with the cylindrical surface is a common helix.

Let AB be the axis of the cylinder, C the center of a right section which intersects the helix OH at O, Q any point on OH, RT the corresponding position of the generatrix piercing the plane ABS, determined by the axis and O, at P, and meeting the plane of the right section at T.

Then P is any point of the meridian section of the helicoid and T any point of the transverse section.

A. To Find the Locus of P.

The plane through the generatrix RT and tangent to the cylinder cuts CO produced at E, ET being tangent to the circular section at D.

PE is parallel to QD which is an element of the cylindrical surface.

Draw through Q a parallel to DE, meeting PE at L.

Draw CO and CD.

Denote the angle COD by θ, the radius of the cylinder by a, the pitch of the helix by P, and the angle PTE by α.

Taking CO produced as the X-axis, and OY, parallel to AB, as the Y-axis, $EO=x$, and $PE=y$.

$x=EC-OC=a(\sec\theta-1)....(1)$.

$y=PL+QD$.

$PL=LQ\tan\alpha=ED\tan\alpha=a\tan\theta\tan\alpha=ca\tan\theta$, letting $\tan\alpha=c$.

$$QD:P=\theta:2\pi;\ QD=\frac{\theta}{2\pi}P=K\theta,$$

where $K=\dfrac{P}{2\pi}$.

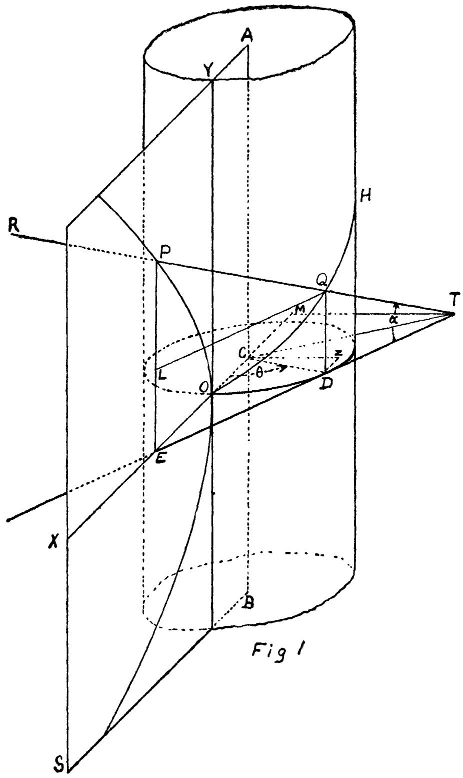


Fig 1

Therefore, $y=K\theta+ca\tan\theta....(2)$.

(1) and (2) are the equations of the meridian section of the helicoid.

There are certain properties of this curve independent of the value of c. These are deduced first. Afterward, the variations in form and the singularities are noted as c varies from $+\infty$ to $-\infty$.

Changing the sign of θ does not affect x but gives to y values, equal numerically, but of contrary sign. Hence, the curve is symmetrical with reference to the X-axis.

Since when $\theta=0$, $y=x=0$, the curve passes through the origin.

When $\theta=\pm\frac{1}{2}\pi$, $x=\infty$, and y becomes infinite except when $c=0$. There are, therefore, infinite branches.

$$\frac{dy}{dx}=\frac{K+ca\sec^2\theta}{a\sec\theta\tan\theta}=\frac{K\cos^2\theta+ca}{a\sin\theta}....(3).$$

When $\theta=\frac{1}{2}\pi$, $\dfrac{dy}{dx}=c....(4)$, and the Y-intercept of the corresponding tangent $=\dfrac{\pi K}{2}+ca$.

An asymptote is $y=cx+\frac{1}{4}P+ca....(5)$, a line crossing AB at a distance

above C equal to one-fourth the pitch and making an angle with the X-axis equal to α.

Equating to zero, $\frac{1}{a}(K\cos^2\theta-2K-ca)\cot\theta\csc\theta$, the derivative of $\frac{dy}{dx}$ with regard to θ, gives

$$\cos\theta=\pm\sqrt{2+\frac{ca}{K}}\ldots.(6).$$

(a). $c>0$.

From (3), when $\theta=0$, $\frac{dy}{dx}=\infty$; the curve is tangent to the Y-axis at the origin.

From (6), there is no point of inflexion.

The curve is represented in Fig. 2, positive values of θ giving points above the axis on OP; negative, points below on OW. UV is an asymptote, CU being $\frac{1}{4}P$.

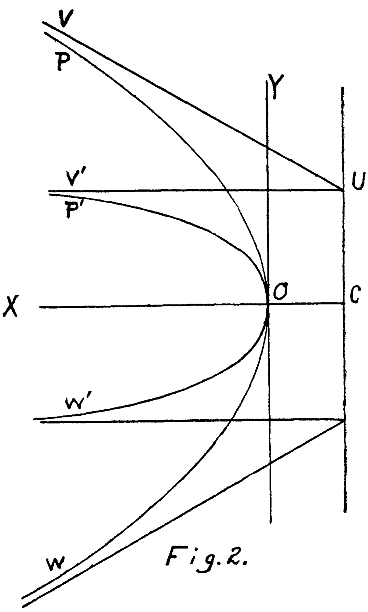


Fig. 2.

(b). $c=0$.

The equation of the asymptote becomes $y=\frac{1}{4}P$, the curve approaching parallellism with the X-axis, and appearing as indicated in Fig. 2 by the accented letters.

(c). $c<0$.

Case I. $c<\frac{K}{a}$, numerically.

It follows from (2) that there is some positive value of θ, less than $\frac{1}{2}\pi$ and other than zero, for which $y=0$. At this point

$$\frac{dy}{dx}=\frac{c}{2}\,\frac{2\theta-\sin2\theta}{\theta.\sin\theta},$$

which is positive when θ is negative, and negative when θ is positive. Hence the point under consideration is a double point of intersection.

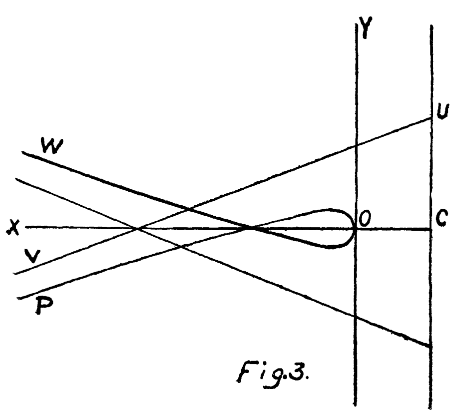


Fig. 3.

Somewhere between it and the origin the curve is parallel to the X-axis, this being the case when $\theta=\cos^{-1}\sqrt{-\frac{ca}{K}}$.

(6) shows that there is no point of inflexion. See Fig. 3.

Case II. $c=\frac{K}{a}$, numerically.

Since $K=-ca$, (2) becomes $y=K(\theta-\tan\theta)$, from which y is negative as θ increases from 0 to $\frac{1}{2}\pi$, and decreases without limit, having neither a maximum nor a minimum value for any value of θ.

When $\theta=0$, (3) assumes the form $\frac{0}{0}$. But it may be written

$$\frac{K(\cos^2\theta-1)}{a\sin\theta}\left(=-\frac{K}{a}\sin\theta\right),$$

which equals zero when $\theta=0$. Hence there is a cusp of the first species at the origin.

(6) gives $\cos\theta=\pm1$, but this does not correspond to a point of inflexion since $\frac{d}{d\theta}\left(\frac{dy}{dx}\right)$ does not change sign as θ passes through 0.

This curve is drawn in Fig. 4.

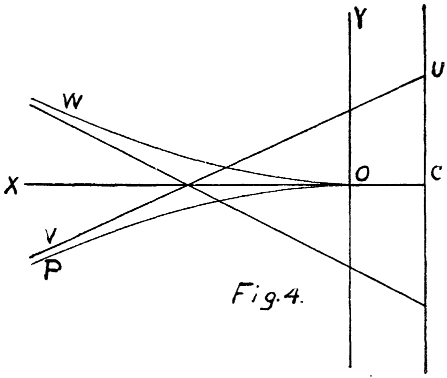


Fig. 4.

Case III. $c>\frac{K}{a}$ but $<\frac{2K}{a}$, numerically.

By (2) y is negative and θ increases from 0 to $\frac{1}{2}\pi$.

By (3) there is no point at which the curve is parallel to the X-axis; at the origin it is tangent to the Y-axis.

By (6) there is a point of inflexion where

$$\theta=\cos^{-1}\sqrt{2+\frac{ca}{K}}.$$

See Fig. 5.

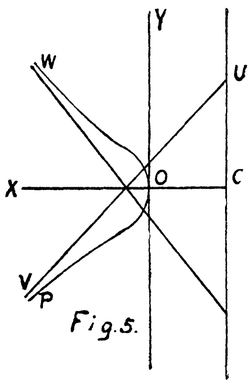


Fig. 5.

Case IV. $c>\frac{2K}{a}$, numerically, but negative.

Points corresponding to θ between 0 and $\frac{1}{2}\pi$ lie below the X-axis. The curve is tangent to the Y-axis at the origin, and, by (6), there is no point of inflexion.

The curve is shown in Fig. 6.

When $c=\infty$, the helicoidal surface becomes the cylindrical, the meridian section coinciding with the Y-axis. As c changes sign, passing through infinity, the portions of the curve, OP and OW, exchange positions with reference to the X-axis.

It may be be noted that in (c) Case II the inclination of the generatrix equals that of the helix, the helicoid becoming the convolute.

Equation (4) shows that the curve at infinity always has the direction of the generatrix when parallel to the meridian plane. This plane remaining fixed, while the inclination of the generatrix varies, the asymptote turns about the stationary point U through 180°, as is evident from equation (5).

B. To Find the Locus of T.

Through T draw a line perpendicular to OC, meeting it, produced, at M. Through C draw a parallel to MT meeting a perpendicular to it from D at Z.

Taking CO and CZ for the X- and Y-axes, respectively, $MC=x$ and $MT=y$.

Since $DT=\frac{QD}{\tan\alpha}=\frac{K\theta}{c}$,

$$x=a\cos\theta-\frac{K\theta}{c}\sin\theta\ldots.(1),$$

$$\text{and } y=a\sin\theta+\frac{K\theta}{c}\cos\theta\ldots.(2),$$

from which

$$x^2+y^2=a^2+\frac{K^2\theta^2}{c^2},\ \text{or, } r^2=a^2+\frac{K^2\theta^2}{c^2},\ (r=CT)\ldots.(3).$$

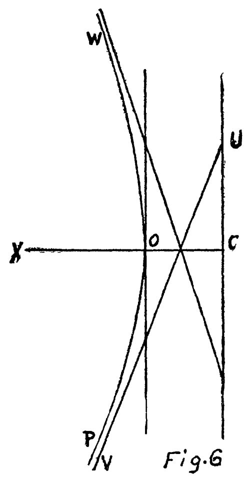


Fig. 6

These are the equations of the transverse section. They show—(1) and (2) together, or (3) alone—that the curve is symmetrical with respect to the X-axis; it is a spiral beginning at O and cutting the axes of X and Y, in general, an infinite number of times at points more and more remote from C.

As in the case of the meridian section, this locus has distinguishing features depending on the value of C. These are shown in Fig. 7, a part of the first spiral only being represented. The loci are traced for values of θ between 0 and π; also, symmetrical curves, OX being the axis of symmetry, for numerically equal negative values of θ.

$$\frac{dy}{dx}=\frac{K\theta\tan\theta-(ca+K)}{K\theta+(ca+K)\tan\theta}=\tan(\theta-\phi)\ldots.(4)$$

where $\phi=\tan^{-1}\frac{ca+K}{K\theta}$.

$$\frac{d}{d\theta}\left(\frac{dy}{dx}\right)=\sec^2(\theta-\phi)\frac{K^2\theta^2+c^2a^2+3caK+2K^2}{K^2\theta^2+(ca+K)^2}\ldots.(5).$$

(*a*). $c>0$.

The curve is perpendicular to the X-axis at O, is concave toward C, and has no point of inflexion. In Fig. 7, OTI is a part of a spiral of this class.

(*b*). $c=0$.

The transverse section coincides with the generatrix when $\theta=0$. In Fig. 7, it is shown as OY', parallel to the Y-axis.

(c). $c<0$.

Case I. $c<\frac{K}{a}$, numerically.

While the X-axis is still crossed at right angles at O, y is negative until θ reaches a value given by the equation

$$\tan\theta+\frac{K}{ca}\theta=0.$$

By (4), the slope is first negative and, by (5) it increases with θ.

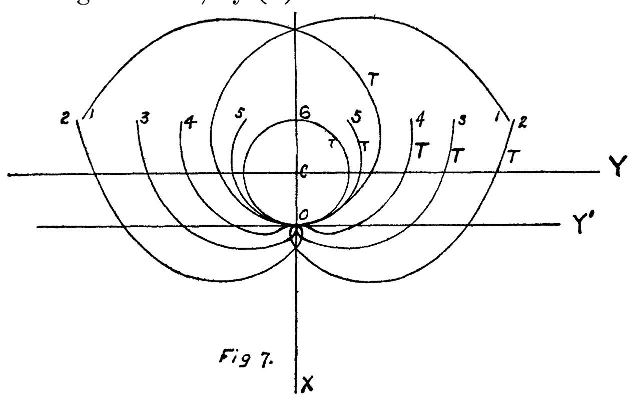


Fig 7.

Hence, there is a double point of intersection as seen in $OT2$.

Case II. $c=\frac{K}{a}$, numerically.

(3) reduces to $r^2=a^2+a^2\theta^2$, the transverse section becoming the involute of the circular section.

(4) gives $\frac{dy}{dx}=\tan\theta$, as might have been inferred from the relation between the curve and its evolute.

When $\theta=0$, $\frac{dy}{dx}=0$, and there is a cusp of the first species at O. There being no point of inflexion, the locus appears as in $OT3$.

Case III. $c>\frac{K}{a}$ but $<\frac{2K}{a}$, numerically.

Again the X-axis is intersected perpendicularly at O.

Equating (5) to zero,

$$\theta=\sqrt{-\left(\frac{ca}{K}+1\right)\left(\frac{ca}{K}+2\right)}.$$

Hence there is a point of inflexion. See $OT4$.

Case IV. $c>\frac{2K}{a}$, numerically, but negative.

There is no point of inflexion. The curve is perpendicular to the X-axis at O and is concave toward C. $OT5$ represents it.

When $c=\infty$, (1) and (2) become $x=a\cos\theta$ and $y=a\sin\theta$, respectively; while (3) reduces to $K^2+y^2=a^2$, or, $r=a$. The transverse section coincides with the right circular section. See $OT6$.

The curve intersects the line OY' for some value of θ less than π when c lies between 0 and $-\frac{2K}{a}$. That this is true may be seen by making $x=a$ in (1), obtaining, after reduction,

$$\tan\tfrac{1}{2}\theta=-\frac{K'}{ca}\,\theta.$$

If $a=0$, (3) becomes $r=\pm\dfrac{K}{c}\theta$, the equation of a spiral of Archimedes.

Under this hypothesis, O and D (Fig. 1) coincide with C.

CT, coinciding with DT, is perpendicular to CD, and, since OCZ is a right angle, $\angle TCZ=\angle OCD=\theta$.

Hence CZ is the initial line and C the pole.

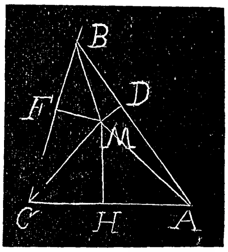
PROVING THE FALSE.

By DR. GEORGE BRUCE HALSTED.

I. THEOREM. Every triangle is isosceles.

HYPOTHESIS. Let ABC be a triangle.

CONCLUSION. $AC=BC$.



PROOF. Take the bisector of $\measuredangle C$, and erect the perpendicular bisector of AB.

From their intersection M drop the perpendiculars MF, MH. Then $\triangle AMD\equiv\triangle BMD$ (by Euclid I. 4).

$\therefore AM=BM$.

But $\triangle CMH\equiv\triangle CMF$ (by Euclid I. 26). $\therefore MF=MH$, and $CF=CH$.

$\therefore \triangle AMH\equiv\triangle BMF$ (by Halsted's *Elements*, 179).

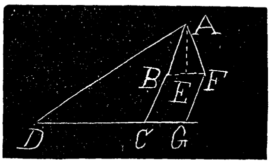
$\therefore BF=AH$. But already we have $FC=HC$.

Therefore $BC=AC$. Q. E. D.

II. Just as false is the following from Olney's *Geometry*, Section VIII, Proposition XIV.

THEOREM. "Two quadrilaterals having three sides of the one equal to the the three corresponding sides of the other, each to each, and the two corresponding angles adjacent to the unknown sides equal, each to each, are equal figures."

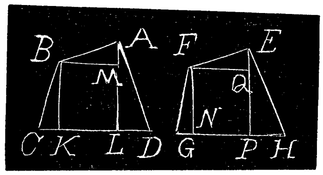
A simple construction, due to my pupil R. L. Moore, shows that they may be as *unequal* as we please.



Make DG greater than DC by whatever sect you please, say a centimeter. Take $\measuredangle DGF=\measuredangle DCB$. Take $GF=CB$. Take A on the perpendicular bisector of BF. Then the quadrilaterals $ABCD$, $AFGD$ fulfill the hypothesis of the theroem.

My friend G. B. M. Zerr, in a moment of distraction, has given the following as proof that these unequals are equal, and it seems that Mr. J. Scheffer agrees therein. See THE AMERICAN MATHEMATICAL MONTHLY, April, 1902, Vol. IX, page 105.

Let $ABCD$, $EFGH$ be the two quadrilaterals; $BC=FG$, $AB=EF$, $AD=EH$, $\measuredangle BCD=\measuredangle FGH$, $\measuredangle ADC=\measuredangle EHG$. Draw BK, AL perpendicular to CD; FN, EP perpendicular to GH; BM perpendicular to AL; FQ perpendicular to EP.



Right triangles BCK and FGN are equal, also right triangles ALD and EPH (by Euclid I. 26).

$\therefore$ $BK=FN$, $AL=EP$, also $AL-BK=AM=EP-FN=EQ$.

$\therefore$ right triangles $ABM=FEQ$; since $AB=FE$ and $AM=EQ$.

$\therefore$ $BM=FQ$. $\therefore$ $BM=KL=FQ=NP$. $\therefore$ $BKLM=FNPQ$.

$\therefore$ $BCK+ADL+ABM+BKLM=FGN+EPH+FEQ+FNPQ$.

$\therefore$ $ABCD=EFGH$, the part equal to the whole. Q. E. D.

Now what is the explanation of these proofs of the false? Not exactly that for the extraordinary blunder which used to stand on page 224 of Wentworth's *Geometry* (for ten years at least from 1877 to 1887), Proposition XIII, §387. "To inscribe a regular polygon of any number of sides in a given circle."

This, like the blunder of McLellan and Dewey, in making number the outcome of measurement, could only coexist with a colossal ignorance of the history of mathematics, while pseudo-proofs like Professor Zerr's, which may be characterized as figure proofs, are given by those who, like Prof. D. E. Smith, need not be supposed unusually ignorant of the history of mathematics.

A figure we should use, but never suppose that we may rest our proof upon it, draw our proof from supposed perception of it. We must always take care that the operations supposed carried out on a figure retain a pure logical validity and inter-connection.

The figure is a help, which, this forgotten, may become a harm.

Moreover, what are called *problems of construction* have a double import. Theoretically, they are really theorems declaring that the existence of certain points, sects, straights, angles, circles, etc., follows logically by rigorous deduction from the existences postulated in our assumptions. Thus the possibility of solving such problems by elementary geometry is a matter absolutely essential in the logical sequence of our theorems.

So, for example, elementary geometry shows (Halsted's *Elements*, §502) that a sect has always trisection points, and this may be expressed by saying we have solved the problem to trisect any sect. Now it happens that a solution of the problem to trisect any angle is impossible with only the assumptions of elementary geometry. Thus any reference to or use of results following from or dependent upon the trisection of the angle would be equivalent to the introduction of additional assumptions or postulates.

Thus the whole "Note on Assumed Constructions" in Beman and Smith's *Geometry*, 1899, page 70, §112, is seen to be as erroneous as Wentworth's problem and Professor Zerr's proof.

Austin, Texas, May 3, 1902.

NOTE. Professor Norris, the proposer of Problem 169, Geometry, over a year ago sent with the supposed theorem a note calling attention to the fact that it was not true in

general. This note got lost, and when the selection of proofs was made, the fact of the theorem being false had slipped my mind.

We are glad that Dr. Halsted has contributed this article, as it will help to keep geometricians on the straight road to rigorous demonstrations. His article on "Some Fallacies in Wentworth's Geometry" has awakened quite an interest among teachers in the high schools of the country.

By his fearless exposition of fallacies in the teaching of Elementary Geometry, Dr. Halstead has done more to improve geometrical instruction in America, perhaps, than any other American mathematician.

However, we do not entirely agree with him in his criticism on the "Note on Assumed Constructions" in Beman and Smith's Geometry, Second Edition. For the benefit of those of our readers who may not have a copy of the book at hand, we quote the note in full as follows:

"It has been assumed, up to Proposition XXVIII, that all constructions were made as required for the theorems. Thus an equilateral triangle has been frequently mentioned, although the method of constructing one has not been indicated; a regular heptagon has been mentioned in Ex. 93, and reference might be made to certain results following from the trisection of an angle, although the solutions of the problems, to construct a regular heptagon and to trisect an angle, are impossible by elementary geometry. But the possibility of solving such problems has nothing to do with the logical sequence of the theorems. One may know that each angle of a regular heptagon is 5-7x180°, whether the regular heptagon admits of construction or not. Nevertheless, an important part of geometry concerns itself with the construction of certain figures—a part of most practical value and of much interest to the student of mathematics."

We can see nothing erroneous in this note. If it is proved by perfectly rigorous and elementary methods that the sum of the interior angles of any convex polygon of n sides is $(n—2)$ straight angles, it certainly follows that each angle of a regular heptagon is 5-7 of a straight angle, whether the heptagon can be constructed by elementary methods or not; or whether, indeed, a regular heptagon can be constructed by any method whatever. So, too, if we are given an angle of 33°, $\frac{1}{3}$ of that angle is 11°, whether the angle can be trisected or not. In this we agree with Professors Beman and Smith. But on the other hand we agree with Dr. Halsted in protesting against requiring constructions of such Propositions as the one referred to in Wentworth's Geometry, viz: "To inscribe a regular polygon of any number of sides in a given circle." Such propositions are misleading alike to teacher and student. We do not remember of having come to us, a student having completed his plane geometry elsewhere who had the slightest suspicion that it is any more difficult to inscribe a regular heptagon in a circle than it is to inscribe a pentagon. In looking through some of the geometries in our possession, we find a few others beside Wentworth who have to plead guilty to this fault. Wentworth has left the proposition referred to out of the last edition of his geometry.

Now as to problem 169, taken from Olney's Geometry, Section VIII. We find on consulting Olney's Geometry, University Edition, 1881, that the theorem as given by Prof. Olney in this edition was not correctly quoted. The theorem, in part, reads as follows: *Two quadrilaterals are equal when the following parts are equal, each to each, in both quadrilaterals, and similarly arranged:* 1.... 2.... 3.... 4. *The three sides and two included angles.* This theorem is definite.

In Olney's Elementary Geometry, edition of 1883, the following proposition is given on page 144: Proposition XII. A quadrilateral is determined when there are given in their order: 1.... 2.... 3. 1st, Three sides and two included angles. 2d, When the two angles are not both included between the known sides, the case may be ambiguous.

In discussing this second case, Olney draws a figure exactly as Dr. Halsted has, thus showing that the proposition is not generally true. We are not aware that this proposition is given in *any* geometry, as published in the MONTHLY.—EDITOR.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

156. Proposed by JAMES F. LAWRENCE. A. B., Professor of Mathematics, Rogers Academy, Rogers, Ark.

Suppose that in a meadow the grass is of uniform quality and growth, and that 6 oxen or 10 colts could eat up 3 acres of pasture in 18-25 of the time in which 10 oxen and 6 colts could eat up 8 acres; or that 600 sheep would require 2 6-7 weeks longer than 660 sheep to eat 9 acres. In what time could 1 ox, 1 colt and 1 sheep eat up 1 acre of pasture, on the supposition that 588 sheep eat as much in a week as 6 oxen and 11 colts?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let the standing grass on one acre be denoted by 1, and the weekly growth on one acre by u. From the first condition: Because 6 oxen eat all that grows on 3 acres each week and in addition $25/18t$ part of the grass standing on 3 acres; and because 1 colt eats $\frac{3}{5}$ times as much as 1 ox, we have what 1 ox eats in 1 week$=\frac{1}{6}(3\times\frac{25}{18t}+3u)=\frac{1}{36t}(25+18tu)$....(1), what 1 colt eats in 1 week $=\frac{1}{60t}(25+18tu)$....(2).

From the second condition we have: What 10 oxen and 6 colts eat in 1 week$=8/t+8u$....(3). We have considered $\frac{18}{25}t$ weeks in first condition, and t weeks in second condition as the time required.

From (1), (2), (3), we have $\frac{1}{90t}(25+9)(25+18tu)=\frac{8}{t}(1+tu)$, or $(25+9)(25+18tu)=720(1+tu)=34(25+18tu)$....(4).

Suppose that 9 acres will support 660 sheep for x weeks; then 1 sheep in 1 week will eat $\frac{3}{220x}(1+ux)$....(5). Also from first statement, 1 sheep will eat $\frac{1}{600}(\frac{9}{x+2\frac{6}{7}}+9u)=\frac{3}{200}(\frac{7}{7x+20}+u)$.... (6).

Now 3 oxen=5 colts=140 sheep.

From (1) and (5), $11x(25+18ut)=252t(1+ux)$....(7).

From (5) and (6), $7ux^2+20ux+7x=200$....(8).

From (4), $65=54tu$, or $u=65154t$. (7) becomes, by substituting u, $5x=6t$, or $t=5x/6$.

$\therefore u=13/9x$. This in (8) gives $x=10$ weeks.

$\therefore u=\frac{13}{90}$, $t=\frac{25}{3}$.

From (1), 1 ox eats in 1 week, $\frac{7}{45}$;

From (2), 1 colt eats in 1 week, $\frac{7}{75}$;

From (5), 1 sheep eats in 1 week, $\frac{1}{300}$.

$\frac{7}{45}+\frac{7}{75}+\frac{1}{300}=\frac{227}{900}$, what 1 ox, 1 colt and 1 sheep eat in 1 week.

The grass on 1 acre in 1 week with what grows$=1+\frac{13}{90}=\frac{103}{90}$.
$\frac{227}{900}-\frac{13}{90}=\frac{97}{900}$, the amount of the standing grass eaten in 1 week.
$1\div\frac{97}{900}=\frac{900}{97}=9\frac{27}{97}$ weeks, the time required.

SOME INTERESTING RULES IN MULTIPLICATION.

BY MARY M. CURRIER, WENTWORTH, N. H.

RULE 1. To multiply one number by another, the multiplier consisting of two digits of which the left-hand digit is 1:

Multiply each figure of the multiplicand by the right-hand figure of the multiplier, and to each product add the figure of the multiplicand following the one multiplied.

Example 1. Multiply 1675 by 13.

$$1675\times13=21775$$

$$\begin{array}{ll}
5\times3= & \quad\quad\quad 15 \\
7\times3=21, & 21+1+5=27 \\
6\times3=18, & 18+2+7=27 \\
1\times3=3, & \;\;3+2+6=11 \\
 & \quad\;\; 1+1=\;2
\end{array}$$

Taking the digits in the units place in these several products beginning with the last, we have, $21775=13\times1675$.

Example 2. Multiply 40928 by 17.

$$\begin{array}{ll}
8\times7= & \quad\quad\quad 56 \\
2\times7=14, & 14+5+8=27 \\
9\times7=63, & 63+2+2=67 \\
0\times7=\;0, & \;\;0+6+9=15 \\
4\times7=28, & 28+1+0=29 \\
 & \quad\;\; 2+4=\;6
\end{array}$$

$\therefore$ $695776=17\times40926$.

RULE 2. To multiply one number by another, the multiplier consisting of three digits, of which the two at the left are ones:

Multiply each figure of the multiplicand by the right hand figure of the multiplier and to each product add the two figures following the one multiplied and the digit in the tens place of the preceding product.

Example 1. Multiply 340726 by 114.

$$\begin{array}{ll}
6\times4= & \quad\quad\quad\quad\;\; 24 \\
2\times4=\;8, & \;\;8+2+6\quad=16 \\
7\times4=28, & 28+1+2+6=37 \\
0\times4=\;0, & \;\;0+3+7+2=12 \\
4\times4=16, & 16+1+0+7=24 \\
3\times4=12, & 12+2+4+0=18 \\
 & \quad\;\; 1+3+4=\;8 \\
 & \quad\quad\quad\;\; 3=\;3
\end{array}$$

$\therefore$ $38842764=114\times340726$.

RULE 3. To multiply one number by another, the multiplier consisting of two digits of which the left hand digit is 2.

Multiply each figure of the multiplicand by the right-hand figure of the multiplier and to each product add twice the figures of the multiplicand following the one multiplied and the digit in tens place of the preciding product.

Example 1. Multiply 3495013 by 26.

$$\begin{array}{ll} 3\times6 & =18 \\ 1\times6=\ 6, & 6+1+2\times3=13 \\ 0\times6=\ 0, & 0+1+2\times1=\ 3 \\ 5\times6=30, & 30+0+2\times0=30 \\ 9\times6=54, & 54+3+2\times5=67 \\ 4\times6=24, & 24+6+2\times9=48 \\ 3\times6=18, & 18+4+2\times4=30 \\ & 3+2\times3=\ 9 \end{array}$$

$\therefore$ 90870338=26$\times$3495013.

From these three rules one may see how the principle may be carried further.

NOTE. We deem these rules of sufficient interest to merit publication in this department. It may be seen that they are results of a compact arrangement of the following method: Multiply 3492 by 117.

$$\begin{array}{rl} 3492 & 2\times7 \qquad\qquad =14 \\ 117 & 9\times7=63,\ 63+1+2 \quad =66 \\ \hline & 4\times7=28,\ 28+6+9+2=45 \\ 14 & 3\times7=21,\ 21+4+4+9=38 \\ 63. & 3+3+4=10 \\ 28.. & 1+\ 3=4 \\ 21... & \\ 3492. & \therefore\ 408564=117\times3492. \\ 3492.. & \\ \hline 408564 & \end{array}$$

EDITOR.

ALGEBRA.

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139. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Solve neatly $\sqrt[4]{(m-x)}=\sqrt[4]{n}-\sqrt[4]{x}$.

Solution by J. H. DRUMMOND, LL. D., Portland, Me.

$(m-x)^{\frac{1}{4}}=n^{\frac{1}{4}}-x^{\frac{1}{4}}\ \ldots.(1)$.

Put $n=p^4$ and $x=y^4$. Substituting these values in (1), raising to the fourth power, and reducing, we have

$$y^4-2py^3+3p^2y^2-2py^3=\frac{m-p^4}{2}.$$

By inspection we see that adding p^4 to the first member it becomes a perfect square. Adding p^4 to both members and extracting the square root we have

$$y^2-py+p^2=\pm\sqrt{\frac{m+p^4}{2}};\text{ then } y=\frac{p\pm[-3p^2\pm2\sqrt{(2m+2p^4)}]^{\frac{1}{2}}}{2}.$$

Restoring values of y and p, we have

$$x=\left(\frac{n^{\frac{1}{4}}\pm[-3n^{\frac{1}{2}}\pm2\sqrt{(2m+2n)}]^{\frac{1}{2}}}{16}\right)^4.$$

It is a pretty good question in Diophantine Analysis to give such values to m and n as will make x a rational whole number. If $m=17$, and $n=81$, $x=16$ or 1. But if we go back to the original equation, it becomes very easy for n and x may be any fourth powers, say p^4 and q^4, and $m=(p-q)^4+q^4$.

Solved similarly by *G. B. M. ZERR*. F. P. Matz solved it by making $x=m\sin\phi$.

140. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

A man pays monthly \$24.50 for 8 years for a loan of \$1250. What is the rate %?

Solution by the PROPOSER.

Let $12r$=rate %.

$$\therefore\ 24.50=\frac{1250r(1+r)^{96}}{(1+r)^{96}-1}.$$

$\therefore\ (2500r-49)(1+r)^{96}=49.\quad \therefore\ r=.02203$ nearly. $12r=26.43\%$.

141. Proposed by JOSEPH V. COLLINS, Ph. D., Professor of Mathematics. State Normal School, Stevens Point. Wis.

How many teams of two horses each can a livery stable man send out who has 10 horses, assuming (1) that we consider the way the team is hitched and (2) that we do not.

Suppose he has 8 horses; 10 horses. Suppose he has 7 buggies, then how many different rigs can he send out, assuming that he has 10 horses, and counting both one and two horse rigs?

No solution of this problem has yet been received.

142. Proposed by A. H. BELL, Hillsboro, Ill.

If x/y is the convergent preceding the complete quotient $(\sqrt{A}+m)/n$; prove that $x^2-Ay^2=\pm n$.

Solution by H. S. VANDIVER, Bala, Pa.

Expand $\sqrt{A}$ in a continued fraction. Let P_k/Q_k denote the convergent preceding $\frac{\sqrt{A}+m}{n}$, and let $\frac{P_{k-1}}{Q_{k-1}}$ denote the convergent immediately preceding P_k/Q_k, then

$$\sqrt{A}=\frac{P_kx_k+P_{k-1}}{Q_kx_k+Q_{n-1}}\text{ where }x_k=\frac{\sqrt{A}+m}{n}.$$

Substituting this value of x_k, ahd simplifying,

$$\sqrt{A}=\frac{P_n(\sqrt{A}+m)+kP_{n-1}}{Q_n(\sqrt{A}+m)+nQ_{n-1}}.$$

Multiplying out, and equating rational and irrational parts, there is obtained

$$Q_k m+nQ_{k-1}=P_k\ldots.(1),$$
$$P_k m+nP_{k-1}=Q_k A\ldots.(2).$$

Solving for n in (2) and making use of the relation $P_kQ_{k-1}-Q_kP_{k-1}=(-1)^k$ we get, $P_k^2-AQ_k^2=(-1)^kn$.

Also solved by *G. B. M. ZERR*.

143. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Solve $x+y+z+a=a\ldots.(1)$.
$x^2+y^2+z^2+u^2=b\ldots.(2)$.
$x^3+y^3+z^3+u^3=c\ldots.(3)$.
$x^4+y^4+z^4+u^4=d\ldots.(4)$.

Solution by MARCUS BAKER, Washington, D. C.

Let u, x, y and z be the roots of the equation

$$X^4-AX^3+BX^2-CX+D=(X-u)(X-x)(X-y)(X-z).$$

For X write $1/V$, whence

$$1-AV+BV^2-CV^3+DV^4=(1-uV)(1-xV)(1-yV)(1-zV);$$

whence $\log(1-AV+BV^2-CV^3+DV^4)$

$$\begin{aligned}&=\log(1-uV)+\log(1+xV)+\log(1-yV)+\log(1-zV),\\&=-uV-\tfrac12u^2V^2-\tfrac13u^3V^3-\tfrac14u^4V^4-\ldots.\\&\quad-xV-\tfrac12x^2V^2-\tfrac13x^3V^3-\tfrac14x^4V^4-\ldots.\\&\quad-yV-\tfrac12y^2V^2-\tfrac13y^3V^3-\tfrac14y^4V^4-\ldots.\\&\quad-zV-\tfrac12z^2V^2-\tfrac13z^3V^3-\tfrac14z^4V^4-\ldots.\\&=-aV-\tfrac12bV^2-\tfrac13cV^3-\tfrac14dV^4-\ldots.\end{aligned}$$

whence $1-AV+BV^2-CV^3+DV^4=e^{-(aV+\frac12bV^2+\frac13cV^3+\frac14dV^4+\ldots.)}$

Developing the second member of this equation, remembering that

$$e^{-x}=1-x-\frac{x^2}{2!}-\frac{x^3}{3!}-\frac{x^4}{4!}\ldots.$$

we have

$$\begin{array}{l|l|l|l|l}1-AV+BV^2-CV^3+DV^4=1-a & V\ -\tfrac12b & V^3\ -\tfrac13c & V\ -\tfrac14d & V^4-\ldots.\\ & \quad+\tfrac12a^2 & \quad+\tfrac12ab & \quad+\tfrac13ac & \\ & & \quad-\tfrac16a^3 & \quad+\tfrac18b^2 & \\ & & & \quad-\tfrac14a^2b & \\ & & & \quad+\tfrac1{24}a^4 & \end{array}$$

Equating like coefficients, we have

$A=a$; $B=\frac12(a^2-b)$; $C+\frac16(a^3-3ab+2c)$; $D=\frac1{24}[a^4-6a^2b+(3b^2+8ac)-6d]$.

Therefore u, x, y and z are the roots of the equations

$$X^4-aX^3+\tfrac{1}{2}(a^2-b)X^2-\tfrac{1}{6}(a^3-3ab+2c)X+\tfrac{1}{24}(a^4-6a^2b[8ac+3b^2]-6d)=0.$$

The *method* here used I obtained many years ago from my old friend James Main of the Coast and Geodetic Survey who died in Washington November 23, 1894, aged 84 years. This method is *general* applying to a set of n equations containing n unkown quantities.

If we have

$$x+y+z+\ldots.=a$$
$$x^2+y^2+z^2+\ldots.=b$$
$$x^3+y^3+z^3+\ldots.=c$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$x^n+y^n+z^n+\ldots.=k$$

then are x, y, z, etc., the roots of

$$X^n-AX^{n-1}+BX^{n-2}-CX^{n-3}+\ldots.\pm K=0,$$

where

$A=a$

$B=(1/2!)\ [a^2-b]$

$C=(1/3!)\ [a^3-3ab+2c]$

$D=(1/4!)\ [a^4-6a^2b+(8ac+3b^2)-6d]$

$E=(1/5!)\ [a^5-10a^3b+5a(4ac+3b^2)-10(3ad+2bc)+24e]$

$F=(1/6!)\ [a^6-15a^4b+5a^2(8ac+9b^2)-15(6a^2d+8abc+b^3)+2(72ae+45bd+20c^2)-120f]$

$G=(1/7!)\ [a^7-21a^5b+35a^3(2ac+3b^2)-105a(2a^2d+3bc+b^3+b^2c)+14(24a^2e+20ac^2+15b^2c+45abd)-84(10af+6be+5cd)+720g]$

Solved in an excellent manner by *G. B. M. ZERR.*

144. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy, Defiance College, Defiance, Ohio.

Show that the number of ways in which 15 different problems may be distributed among 5 students so that each student shall have three of them, is $N=(5.3)!/(3!)^5$.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and M. E. GRABER, Heidelberg University, Tiffin, Ohio.

It is demonstrated that the number of ways $a+b+c+\ldots.$ can be divided into groups containing a things in one group, b in another, etc., is

$$\frac{[a+b+c+\ldots.]!}{[a!][b!][c!]\ldots.}.$$

Let $a=b=c=$ etc. $=3$.

$\therefore\ n=\dfrac{[5.3]!}{[3!]^5}=\dfrac{15!}{[3!]^5}$.

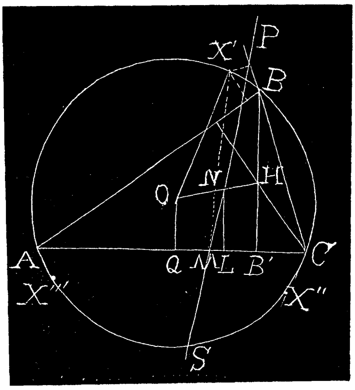
GEOMETRY.

172. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud. Gloucestershire, England.

The center N of the 9-point circle of a triangle ABC lies on P, the pedal line of a point on the circumcircle. Find the angle of intersection of P and AB.

Solution by MARCUS BAKER, Washington. D. C.

In the annexed figure PS is *Simpson's line*, or the pedal line of the circumcircle. Lines drawn from any point X' in the circumcircle perpendicular to the sides of ABC determine three collinear points on Simpson's line. In the figure a point X has been so chosen that the resulting *Simpson line* passes through N, the center of the Twelve Point or Feuerbach circle. H is the orthocenter, O the circumcenter, and N the Feuerbach center. $NO=NH=\frac{1}{2}OH$.



Let $\varphi=$ the angle NML to be determined.

$$\tan\varphi=\frac{LN}{LM}=\frac{\frac{1}{2}[OQ+HB']}{\frac{1}{2}QB'-QM}$$

$$=\frac{\frac{1}{2}[R\cos B+2R\cos C\cos A]}{\frac{1}{2}[2R\sin C\cos A-R\sin B]-R\cos\varphi}$$

$$\text{or }\tan\varphi=\frac{\cos C\cos A+\frac{1}{2}\cos B}{\sin C\cos A-\frac{1}{2}\sin B-\cos\varphi}$$

$$=\frac{m}{n-\cos\varphi}\ldots.[1],$$

where $m=\cos C\cos A+\frac{1}{2}\cos B$, $n=\sin C\cos A-\frac{1}{2}\sin B$.

From [1], $n-\dfrac{m}{\tan\varphi}=\cos\varphi$, whence $n^2-\dfrac{2mn}{\tan\varphi}+\dfrac{m^2}{\tan^2\varphi}=\dfrac{1}{1+\tan^2\varphi}$

which reduces to

$$\tan^4\varphi-2\frac{m}{n}\tan^3\varphi+\frac{m^2+n^2-1}{n^2}\tan^2\varphi-2\frac{m}{n}\tan\varphi+\frac{m^2}{n^2}=0\ldots.[2].$$

This equation gives one (or more?) lines filling the conditions. By permuting the letters in [1] two more are similarly found. Points X'' and X''' of the figure determine two more lines fulfilling the conditions.

Also demonstrated by *G. B. M. ZERR.*

173. Proposed by P. C. CULLEN, Principal of Schools, Indianola, Neb.

To construct a circle tangent to a given line at a given point such that tangents drawn to this circle and passing through two fixed points shall be parallel.

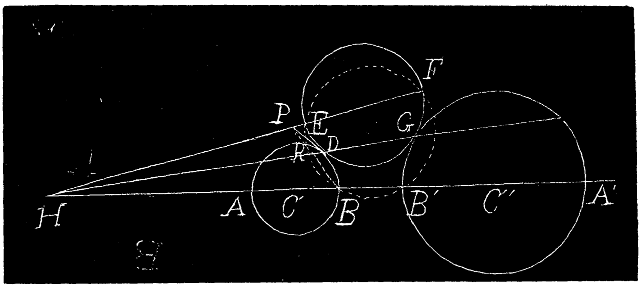
No solution of this problem has been received.

174. Proposed by J.M.HOWIE, Professor of Mathematics, The Nebraska State Normal School, Peru, Neb.

Describe a circle which shall pass through a given point and be tangent to two given circles.

Solution by G. B. M. ZERR, A. M., Ph. D.. The Temple College, Philadelphia, Pa., and M. E. GRABER, Heidelberg University, Tiffin, Ohio.

Let C, C' be the centers of the given circles; E the given point. Describe a circle through EBB' and let H be the homothetic center of C, C'. Draw FEH and $B'BH$. Let the circle through $B'BE$ cut C again in R. Draw BRP and from P draw PD tangent to C, then the circle circumscribing the triangle FED is the circle required, for $HD.HG=HB'.HB$. Since there can be two tangents drawn from P to C, there can be drawn two circles tangent to C, C' so that the line joining their points of contact shall pass through H. Similarly, two circles can be drawn tangent to C, C' passing through E so that the line joining their points of contact shall pass through the anti-homothetic center H' lying between C and C'.



Therefore, there are four circles satisfying the condition.

175. Proposed by W. P. WEBBER, Mississippi Normal College, Houston. Miss.

A field is enclosed by a fence in circular form and a straight gate 20 feet wide. The fence is 100 feet in length. How much land is in the field? [Solution by most elementary method possible.]

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College. Philadelphia. Pa.; J. SCHEFFER, A. M., Hagerstown, Md.; and the PROPOSER.

Chord $AB=20$, arc $ADB=100$.

Let $\angle ACB=\theta$, $AC=r$.

Then $[2\pi-\theta]r=100$, $r\sin\frac{1}{2}\theta=10$.

$\therefore$ $2\pi=\theta+10\sin\frac{1}{2}\theta$.

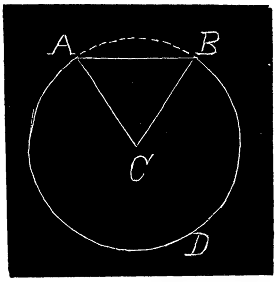
$\therefore$ $\theta=62°\ 32'$ nearly, $r=19.267$ feet.

Area sector $ADB=\dfrac{[360°-62°\ 32']\pi r^2}{360°}$

$=963.64$ square feet.

Area triangle $ACB=164.69$ square feet.

Total area$=1128.33$ square feet.



CALCULUS.

133. Proposed by NELSON L. RORAY, South Jersey Institute, Bridgeton, N. J.

Integrate $\int\frac{\sqrt{[1+y]}}{1+y^2}dy$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let $1+y=z^2$.

$$\therefore \int\frac{\sqrt{[1+y]}}{1+y^2}dy=\int\frac{2z^2dz}{1+[z^2-1]^2}=[1+\sqrt{-1}]\int\frac{dz}{1-\sqrt{-1}+z^2\sqrt{-1}}$$

$$+[1-\sqrt{-1}]\int\frac{dz}{1+\sqrt{-1}-z^2\sqrt{-1}}=-[1-\sqrt{-1}]\int\frac{dz}{1+\sqrt{-1}-z^2}$$

$$-[1+\sqrt{-1}]\int\frac{dz}{1-\sqrt{-1}-z^2}=u.$$

Let $1+\sqrt{-1}=a^2$, $1-\sqrt{-1}=b^2$.

$$\therefore u=-\frac{1}{a^3}\int\left[\frac{1}{a+z}+\frac{1}{a-z}\right]dz-\frac{1}{b^3}\int\left[\frac{1}{b+z}+\frac{1}{b-z}\right]dz$$

$$=\frac{1}{a^3}\log\left[\frac{a-z}{a+z}\right]+\frac{1}{b^3}\log\left[\frac{b-z}{b+z}\right]+C.$$

$$\therefore u=\left[\frac{1}{1+\sqrt{-1}}\right]^{\frac{3}{2}}\log\left[\frac{\sqrt{[1+\sqrt{-1}]}-\sqrt{[1+y]}}{\sqrt{[1+\sqrt{-1}]}+\sqrt{[1+y]}}\right]$$

$$+\left[\frac{1}{1-\sqrt{-1}}\right]^{\frac{3}{2}}\log\left[\frac{\sqrt{[1-\sqrt{-1}]}-\sqrt{[1+y]}}{\sqrt{[1-\sqrt{-1}]}+\sqrt{[1+y]}}\right]+C.$$

Also solved by *F. P. MATZ*. An incorrect solution was received from *H. C. WHITAKER*.

134. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

To find the curve for which the sum of that part of the tangent, lying between the point of contact and the axis of abscissas, and the corresponding ordinate is constant$=c$, and which passes through the point (a, b).

Solution by F. P. MATZ, Sc. D., Ph. D., Defiance College, Defiance, O.; G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; COOPER D. SCHMIDT, A. M., University of Tennessee, Knoxville, Tenn.; and the PROPOSER.

According to the conditions of the problem, *Ordinate*+*Tangent*=*Constant*.

That is, $y+y\sqrt{\left(\frac{dx}{dy}\right)^2+1}=C\ldots.[1]$.

Therefore, $dx=\dfrac{\sqrt{[C^2-2Cy]}}{y}dy$....[2].

or, $x=2\sqrt{[C^2-2Cy]}+C\log\left[\dfrac{\sqrt{[C^2-2Cy]}-C}{\sqrt{[C^2-2Cy]}+C}\right]+C'$....[3].

For $x=a$ and $y=b$, as per the problem, [3] gives the required equation

$$x-a=2\{\sqrt{[C^2-2Cy]}-\sqrt{[C^2-2bC]}\}$$

$$+C\log\left[\frac{\sqrt{[C^2-2Cy]}-C}{\sqrt{[C^2-2Cy]}+C}\Big/\frac{\sqrt{[C^2-2bC]}-C}{\sqrt{[C^2-2bC]}+C}\right]....[4].$$

Also solved by *L. C. WALKER* and *H. C. WHITAKER*. Professor Matz gave a second solution using polar co-ordinates.

135. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

To find the equation of the evolute of the common catenary

$$y=(\tfrac{1}{2}c)(e^{x/c}+e^{-x/c}).$$

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and M. E. GRABER, Heidelberg University, Tiffin, O.

$$\frac{dy}{dx}=\tfrac{1}{2}(e^{x/c}-e^{-x/c}),\quad \frac{d^2y}{dx^2}=\frac{1}{2c}(e^{x/c}+e^{-x/c}).$$

$\therefore$ $x-m+(y-n)dy/dx=0$, $1+(dy/dx)^2+(y-n)d^2y/dx^2=0$ become

$$x-m+\tfrac{1}{4}c(e^{2x/c}-e^{-2x/c})-\tfrac{1}{2}n(e^{x/c}-e^{-x/c})=0....(1).$$

$$1+\tfrac{1}{4}(e^{x/c}-e^{-x/c})^2+\tfrac{1}{4}(e^{x/c}+e^{-x/c})^2-\frac{n}{2c}(e^{x/c}+e^{-x/c})=0....(2).$$

From (2), $e^{4x/c}-\dfrac{n}{c}e^{3x/c}+2e^{2x/c}-\dfrac{n}{c}e^{x/c}+1=0$.

Let $e^{x/c}=z$.

$\therefore$ $z^4-\dfrac{n}{c}z^3+2z^2-\dfrac{n}{c}z+1=0$, $(z^2+1)(z^2-\dfrac{n}{c}z+1)=0$.

$\therefore$ $z=\dfrac{n\pm\sqrt{(n^2-4c^2)}}{2c}=e^{x/c}$ and $\dfrac{n\mp\sqrt{(n^2-4c^2)}}{2c}=e^{-x/c}$.

$\therefore$ $x=c\log\left(\dfrac{n\pm\sqrt{(n^2-4c^2)}}{2c}\right)$.

These values in (1) give us

$$c\log\left(\frac{n\pm\surd(n^2-4c^2)}{2c}\right)-m\mp\frac{n}{4c}\surd(n^2-4c^2)=0,$$

for the equation to the evolute.

136. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Evaluate the definite integral

$$\int_0^1\int_0^1\frac{v^{l-1}u^{m-1}(1-v^n)^{p-1}(1-u^s)^{r-1}dvdu}{[bv^n+c(1-v^n)]^{p+l/n}(u^s+a)^{r+m/s}}.$$

Solution by the PROPOSER.

Let $u^n=z$, $u^s=w$, A=value of integral.

$$\therefore A=\int_0^1\int_0^1\frac{z^{l/n-1}w^{m/s-1}(1-z)^{p-1}(1-w)^{r-1}dzdw}{[bz+c(1-z)]^{p+l/n}(w+a)^{r+m/s}}.$$

Let $z=\dfrac{cy}{b(1-y)+cy}$, $\quad\dfrac{w}{w+a}=\dfrac{x}{1+a}$.

$$\therefore A=\frac{1}{a^r(1+a)^{m/s}b^{l/n}c^p}\int_0^1\int_0^1 y^{l/n-1}x^{m/s-1}(1-y)^{p-1}(1-x)^{r-1}dydx$$

$$=\frac{1}{a^r(1+a)^{m/s}b^{l/n}c^p}\cdot\frac{\Gamma(l/n)\Gamma(p)\Gamma(m/s)\Gamma(r)}{\Gamma(l/n+p)\Gamma(m/s+r)}.$$

MECHANICS.

133. Proposed by J. C. CORBIN, Superintendent of Schools, Pine Bluff, Ark.

A stick of square-edged timber is 20 feet long, 10 inches square at large end, and 6 inches square at small end. How far from either end must a hand spike be placed so that two men with the hand spike and one man at the end shall each have an equal weight to carry?

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and P. H. PHILBRICK, C. E., Lake Charles, La.

Let H=height of pyramid, base 10 inches square; h=height of pyramid, base 6 inches square; z, z_1, z_2 the distances of the centers of mass of the frustum, large pyramid and small pyramid, respectively, from the larger base; m, n the masses of the two pyramids.

$\therefore mz_1=nz_2+(m-n)z$, $z_1=\frac{1}{4}H$, $z_2=\frac{1}{4}h+H-h=H-\frac{3}{4}h$.

Also the masses are to each other as the cubes of the heights.

$\therefore \frac{1}{4}H^4=h^3(H-\frac{3}{4}h)+z(H^3-h^3)$.

$$\therefore z=\frac{H-h}{4}\left(\frac{H^2+2Hh+3h^2}{H^2+Hh+h^2}\right).$$

Let A, a be the lower and upper bases.

$$\therefore z=\tfrac{1}{4}(H-h)\left(\frac{A+2\sqrt{(Aa)}+3a}{A+\sqrt{(Aa)}+a}\right).$$

$H-h=20$ feet, $A=\frac{100}{144}=\frac{25}{36}$ square feet, $a=\frac{36}{144}=\frac{1}{4}$ square foot.

$\therefore z=8\frac{18}{49}$ feet. $20-8\frac{18}{49}=11\frac{31}{49}$.

Taking moments about the center of mass, $2x=11\frac{31}{49}$, $x=5\frac{40}{49}$, $8\frac{18}{49}-5\frac{40}{49}=2\frac{27}{49}$ feet from the larger end.

$2y=8\frac{18}{49}$, $y=4\frac{9}{49}$, $11\frac{31}{49}-4\frac{9}{49}=7\frac{22}{49}$ feet from the smaller end.

134. Proposed by G. B. M. ZERR, A. M., Ph. D., Profeseor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

If $pv=Rt-b/tv$ be the equation for CO_2 gas, find the total, external and internal work done in compressing the gas from 102 to 136 atmospheres at a constant temperature 16°C, and constant volume, $R=63.23$, $b=481600$ for CO_2.

Solution by the PROPOSER.

136 atmospheres $=2000$ lbs. $=p_2$, 102 atmospheres $=1500$ lbs. $=p_1$.

$$pv=Rt-b/tv, \text{ or } v=\frac{Rt}{2p}+\frac{1}{2p}\sqrt{\frac{R^2t^3-4bp}{t}},\quad \frac{dv}{dt}=\frac{R}{2p}+\frac{R^2t^3+2bp}{2pt\sqrt{(R^2t^4-4bpt}}$$

$$=\frac{R}{2p}+\frac{b}{t\sqrt{(R^2t^4-4bpt)}}+\frac{R^2t^2}{2p\sqrt{(R^2t^4-4bpt)}}.$$

$$\text{External work}=t\int_{p_1}^{p_2}\left(\frac{dv}{dt}\right)dp$$

$$=t\int_{p_1}^{p_2}\left(\frac{R}{2p}+\frac{b}{t\sqrt{(R^2t^4-4bpt)}}+\frac{R^2t^2}{2p\sqrt{(R^2t^4-4bpt)}}\right)dp$$

$$=Rt\log_e\left(\frac{Rt^2-\sqrt{(R^2t^4-4bp_2t)}}{Rt^2-\sqrt{(R^2t^4-4bp_1t)}}\right)+\frac{1}{2t}[\sqrt{(R^2t^4-4bp_1t)}-\sqrt{(R^2t^4-4bp_2t)}].$$

Now $t=273+17=290$, $R=63.23$, $b=481600$.

$\therefore$ External work $=18336.7\log_e(1.3367)+46.087=5367.47$ ft. lbs.

$$\text{Total work}=\int_{p_1}^{p_2}vdp=\int_{p_1}^{p_2}\left(\frac{Rt}{2p}+\frac{1}{2p}\sqrt{\frac{R^2t^3-4bp}{t}}\right)dp$$

$$=Rt\log_e\left(\frac{Rt^2-\sqrt{(R^2t^4-4bp_2t)}}{Rt^2-\sqrt{(R^2t^4-4bp_1t)}}\right)+\frac{1}{t}[\sqrt{(R^2t^4-4bp_2t)}-\sqrt{(R^2t^4-4bp_1t)}].$$

$\therefore$ Total work=5275.3 ft. lbs.

$$\text{Internal work}=\int_{p_1}^{p_2}(v-dv/dt)dp=\frac{3}{2t}[\sqrt{(R^2t^4-4bp_2t)}-\sqrt{(R^2t^4-4bp_1t)}]$$
$$=-138.261 \text{ ft. lbs.}$$

Let γ be the ratio of specific heat at constant pressure to specific heat at constant volume; S=dynamic specific heat at constant volume; u=velocity of sound in CO_2, g=gravity, δ=density of mercury, d=density of CO_2, h=height of barometer.

Then $u=\left(\dfrac{g\delta h[1+(t/273)]\gamma}{d}\right)^{\frac{1}{2}}$. Let $t=0$, then $g=32.2$ feet. $\delta=13.59$, $h=29.92$ inches$=2.4935$ feet, $d=.00198$, u by experiment$=856$ feet.

$\therefore \gamma=du^2/g\delta h=1.3296$.

At 0°C., $S=R/(\gamma-1)=63.23/.3296=191.84$.

For any temperature, t, $S_t=B+t\int_{\infty}^{v}(d^2p/dt^2)dv$.

$dp/dt=R/v+b/t^2v^2$, $d^2p/dt^2=-2b/t^3v^2$.

For $t=273$ absolute temperature and $v=1$ cubic foot, $S_t=191.84=B+2b/t^2v=B+963200/(273)^2$.

$\therefore B=178.943$.

For $t=290°$ absolute or 17°C., $S_t=178.943+963200/(290)^2=190.396$.

$5273.3\div190.396=27.7$°C., the amount the temperature would rise if total work were converted to heat.

135. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

What force acting at an inclination ω with a horizontal line on the center of a wheel of given weight will roll the wheel over an immovable cylindric log whose diameter is $(1/m)$th that of the wheel?

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

If the meaning is to find the force F which will *start* the rolling the following is the solution:

If $\phi=\cos^{-1}\left(\dfrac{2\sqrt{m}}{m+1}\right)$=the angle the radius of the wheel through the point of contact with the log, and $\phi+\omega=\alpha$, the angle included by this radius and the direction of the required force, and W=the weight of the wheel, we have, taking moments about the point of contact,

$$F.r\sin\alpha=W.\frac{2r\sqrt{m}}{m+1}, \text{ giving } F.$$

DIOPHANTINE ANALYSIS.

90. Proposed by H. S. VANDIVER, Bala, Pa.

Prove that it is always possible to find an infinite number of positive integral values of x, y, and z, such that the relation $z^2=x^2+bxy+cy^2$ is satisfied, b and c being any integers whatever.

II. Solution by J. H. DRUMMOND, LL. D., Portland, Me.

Take $y=mx$, and $z=nx$ and reduce, and we have $n^2=1+bm+cm^2=$(say) $(pm-1)^2=p^2m^2-2pm+1$, and

$$m=\frac{2p+b}{p^2-c} \text{ and } n=\frac{p^2+bp+c}{p^2-c}.$$

Take $x=p^2-c$, and we have $y=2p+b$ and $z=p^2+bp+c$, in which b and c may be any number and p any number that will make p^2 greater than c.

91. Proposed by LON C. WALKER. A. M.. Professor of Mathematics. Petaluma High School. Petaluma, Cal.

Find the least three positive integral numbers whose sum, sum of their squares, and sum of their cubes shall each be rational squares.

Solution by the PROPOSER.

Let the required numbers by ax, bx, and cx, respectively. Then, by the conditions of the problem, we have

$$(a+b+c)x=\square \text{....(1)}, \quad (a^2+b^2+c^2)x^2=\square \text{....(2), or}$$
$$a^2+b^2+c^2=\square \text{....(3); and } (a^3+b^3+c^3)x^3=\square \text{....(4), or}$$
$$(a^3+b^3+c^2)x=\square \text{....(5).}$$

Assume $(a+b+c)x=x^2$; then $a+b+c=x$....(6).

Now assume $(a^3+b^3+c^3)x=\alpha^2x^2$; then $x=\dfrac{a^3+b^3+c^3}{\alpha^2}$....(7).

Equating the values of x in (6) and (7), and clearing of fractions,

$$(a+b+c)\alpha^2=a^3+b^3+c^3\text{....(8).}$$

By assuming $\alpha=a-b-c$, substituting in (8), and reducing, we get

$$a^2+(b+c)a=3bc\text{....(9).}$$

From (9), $a=-\frac{1}{2}(b+c)\pm\frac{1}{2}\sqrt{(b^2+14bc+c^2)}$....(10).

Now let $b^2+14bc+c^2=\square=(b-\dfrac{\beta}{\gamma}c)^2$, from which $\dfrac{b}{c}=\dfrac{\beta^2-\gamma^2}{2\beta\gamma+14\gamma^2}$.

Hence put $b=\beta^2-\gamma^2$ and $c=2\beta\gamma+14\gamma^2$; then $a=6\beta\gamma-6\gamma^2$.

By substituting these values in (3), and reducing, we get

$$\beta^4+38\beta^2\gamma^2-16\beta\gamma^3+233\gamma^4=\square \ldots.(11).$$

By putting the left member of $(11)=(\beta^2+19\gamma^2)^2$, and solving, we find $\beta=-8\gamma$. To make β positive, let $\beta=\delta-8\gamma$; then substituting in (11), reducing, and putting the left member$=(\delta^2-16\gamma\delta-83\gamma^2)^2$, we find $\delta=1332\gamma/83$, which is an integer when $\gamma=83$. Then $\delta=1332$, $\beta=668$, $a=291330$, $b=439335$, $c=207334$, and $x=937999$. Hence the required numbers are

$$ax=273267248670.$$
$$bx=412095790665.$$
$$cx=194479084666.$$

PROBLEMS FOR SOLUTION.

ARITHMETIC.

160. **Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College. Defiance, Ohio.**

A farm is rented for $\$R,=\$300$, in cash and a certain number of bushels of wheat. When wheat is $\$n,=\$4\text{-}5$, per bushel the rent is $p\%,=12\frac{1}{2}\%$, lower than when wheat is $\$m,=\$1\ 1\text{-}5$, per bushel. Find the number of bushels of wheat.

161. **Proposed by F. M. SHIELDS, Coopwood, Miss.**

If 1 man, 1 boy and 1 girl catch 1 trout, 1 perch and 1 minnow in 5 minutes, and 1 man, 2 boys and 3 girls catch 1 trout, 2 perch and 3 minnows in 6 minutes, how many minutes will be required for 2 men, 3 boys and 4 girls to catch 5 trout, 11 perch and 17 minnows?

ALGEBRA.

162. **Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy, Defiance College, Defiance, Ohio.**

By Sylvester's dialytic method form the eliminant between $mx^3+py^2=0$ ....(1), and $px^2+my^3=0$....(2). Also between $mx^4+py=0$....(1), and $px^3+my^3=0$....(2).

163. **Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg. Pa.**

Solve $x^4-x=14$, by quadratics.

GEOMETRY.

189. **Proposed by J. C. CORBIN, Pine Bluff, Ark.**

The perpendicular from the right angle on the hypotenuse of a right-angled-triangle is a harmonic mean between the segments of the hypotenuse made by the point of contact of the inscribed circle. [From Casey's Sequel to Euclid.]

190. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

Find the locus of the centers of sections of an ellipsoid by planes which are at a constant distance from the center.

CALCULUS.

154. Proposed by B. R. DOWNER, Hopkinsville, Ky.

At the equinox, when the sun is on the celestial equator, a man starts driving on a perfectly level plain at six o'clock in the morning, and continues, going always from the sun, at the uniform rate of six miles per hour, until six o'clock in the evening. Required the path he will travel and the distance in a straight line from starting point to stopping point.

155. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Solve the differential equations:

(*A*). $\frac{d^4y}{dx^4}+2\frac{d^2y}{dx^2}=\sin 2x+\sin x-x.$ (*B*). $\frac{d^2y}{dx^2}+2\frac{dy}{dx}=\sin 2x+\sin x-x.$

MECHANICS.

144. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Pressure is applied perpendicularly to the plane surface yz, bounding an otherwise infinite isotropic solid. Find the resultant displacements, if the pressure varies as $\sin\left(\frac{2\pi y}{a}\right)+\sinh\left(\frac{2\pi y}{a}\right)$.

145. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

ABCD, *GCEF* are equal parallelograms, *DCG* and *BCE* being straight lines. If the figure be considered as formed of smooth light jointed bars and if *BD* be a light rod, and the whole be suspended from *A*, find the stress in *BD* if a weight be hung from *F*. Also find the stress if a light rod *GE* replace *BD*.

DIOPHANTINE ANALYSIS.

104. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

(1). The cube root of three cube numbers equals the square root of two square numbers. Determine the numbers.

(2). The sum of the square roots of three square numbers equals the sum of the cube roots of three cube numbers. Determine the numbers.

105. Proposed by H. S. VANDIVER, Bala, Pa.

Every odd factor of a^n+b^n is of the form $1(\text{mod}\,2n)$.

AVERAGE AND PROBABILITY.

129. Proposed by J. K. ELLWOOD, Principal of Colfax School, Pittsburg, Qa.

A and B play with two dice, A throwing. If he throws 7 or 11, he wins; if he throws 3, or two aces, or two sixes, B wins. But if he throws 4, 5, 6, 8, 9, or 10, he continues throwing to duplicate this throw, in which event he wins; if in throwing, however, he throws 7, B wins. What is the expectancy of each? [This is the regulation "crap" game, B being banker.]

130. Proposed by L. C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma. Cal.

Four points are taken at random on the surface of a given sphere; show that the average volume of the tetraedron formed by the planes passing through the points taken three and three, is 1-35 of the volume of the given sphere.

NOTES.

Dr. Halsted's article on Non-Euclidean Geometry which was to appear in the March issue of Everybody's Magazine has been unavoidably delayed, so says the editor of that Magazine. But the article is now in type and will soon appear in print.

The mathematicians of the Pacific coast held a meeting in San Francisco on May 3, and organized the second section of the American Mathematical Society, to be known as the Pacific Section. Professor Irving Stringham of California University was elected chairman, and Professor G. A. Miller of Stanford University secretary. The section will hold two meetings per year—in May and December—in or near San Francisco.

The article in the April number of THE MONTHLY on "The Betweenness Assumptions" has called forth some noteworthy comments. Dr .E. H. Moore, of the University of Chicago, writes Dr. Halsted: "I have received from you the April number of THE AMERICAN MATHEMATICAL MONTHLY, containing the proof by Mr. R. L. Moore of the redundancy of Hilbert's Axiom II 4. *The proof is certainly delightfully simple.*" Dr. Moore is so impressed therewith that he has written also to Mr. R. L. Moore: "I read with much interest, the other day, your proof of the redundancy of Hilbert's Axiom II 4, in his system I, II, as exhibited by Professor Halsted in the current number of THE AMERICAN MATHEMATICAL MONTHLY. Today I received from Professor Halsted a copy of that number. This is in response to a letter I sent him a week or so ago stating that I should be pleased to receive for publication in the Transactions the delightfully simple proof of the redundancy of which he wrote to me. I certainly agree with him in this estimate of your proof. * * * I remain with considerable interest in the progress of your mathematical career. Yours very truly; E. H. Moore."

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. IX. JUNE-JULY, 1902. NOS. 6-7.

THE ORDER OF A CERTAIN SENARY LINEAR GROUP.

By PROF. L. E. DICKSON, Ph. D.

In the March number of the MONTHLY, the writer determined the factors of the determinant D of a certain square matrix of order six:

$$(1)\qquad \begin{vmatrix} \mathrm{I} & \alpha & \beta & \gamma & \delta & \varepsilon \\ \beta & \mathrm{I} & \alpha & \delta & \varepsilon & \gamma \\ \alpha & \beta & \mathrm{I} & \varepsilon & \gamma & \delta \\ \gamma & \delta & \varepsilon & \mathrm{I} & \alpha & \beta \\ \delta & \varepsilon & \gamma & \beta & \mathrm{I} & \alpha \\ \varepsilon & \gamma & \delta & \alpha & \beta & \mathrm{I} \end{vmatrix}$$

It is readily shown that the product of two such matrices is a third matrix of the same form. Hence, if we assign to I, α, β, γ, δ, ϵ all sets of values in a given field, such that D does not equal 0, we obtain a set of matrices having the group property. The group may be represented concretely as a linear homogeneous group in six variables. It is proposed to determine the order of this group in the Galois Field of order p^n, designated $GF[p^n]$. We have only to find the number of sets of elements I, ..., ε such that D does not equal 0. It was shown in the MONTHLY that

$$(2)\qquad D=(\mathrm{I}+\alpha+\beta+\gamma+\delta+\epsilon)(\mathrm{I}+\alpha+\beta-\gamma-\delta-\varepsilon)\triangle^2,$$

$$(3)\qquad \triangle\equiv\mathrm{I}^2+\alpha^2+\beta^2-\mathrm{I}\alpha-\mathrm{I}\beta-\alpha\beta-\gamma^2-\delta^2-\varepsilon^2+\gamma\delta+\gamma\varepsilon+\delta\varepsilon.$$

If $p=3$, $\triangle$ is the difference of two squares, so that

$$D\equiv(\mathrm{I}+\alpha+\beta+\gamma+\delta+\varepsilon)^3(\mathrm{I}+\alpha+\beta-\gamma-\delta-\epsilon)^3 \qquad (\mathrm{mod}\,3).$$

For 3^{5n} sets of values of I, ..., ε in the $GF[3^n]$, the first factor vanishes. Similarly for the second factor. Both vanish simultaneously for 3^{4n} sets of values. Hence the number of sets for which D does not equal 0, *i. e.*, the order of G, is $3^{6n}-(2.3^{5n}-3^{4n})=3^{4n}(3^n-1)^2$.

For p not equal to 3, we set $A=\mathrm{I}-\alpha$, $B=\mathrm{I}-\beta$, $C=\gamma-\delta$, $D=\gamma-\varepsilon$.

Then $\triangle=(A^2-AB+B^2)-(C^2-CD+D^2)$, and the two linear factors of D become

$$f\equiv3(\mathrm{I}+\gamma)-(A+B)-(C+D),\ f_1\equiv3(\mathrm{I}-\gamma)-(A+B)+(C+D).$$

The sets I, γ, A, B, C, D for which $D=0$, fall into three classes:

f not equal to 0, f_1 not equal to 0, $\triangle=0$;
f not equal to 0, $f_1=0$; $f=0$,

the second class not occurring if $p=2$, since then $f=f_1$. The third class includes p^{5n} sets. If p is not equal to 2, the second class includes $p^{2n}(p^{3n}-p^{2n})$ sets, since it includes all sets for which

$$3\mathrm{I}-A-B=3\gamma-C-D \text{ not equal to } 0.$$

To determine the number of sets in the first class, let ω be a root of $\omega^2+\omega+1=0$. Since p is not equal to 3, ω^2 is not equal to ω. Then

$$\triangle=(A+\omega B)(A+\omega^2B)-(C+\omega D)(C+\omega^2D).$$

If ω belongs to the $GF[p^n]$, $A+\omega B$ and $A+\omega^2B$ are independent elements of the $GF[p^n]$. Hence there are as many sets A, B, C, D making $\triangle=0$ as there are sets in the $GF[p^n]$ making $xy-zw=0$, viz.,* $p^n(p^{2n}+p^n-1)$.

If ω does not belong to the $GF[p^n]$, we set

$$(4) \qquad A+\omega B=X,\ C+\omega D=Y.$$

Since $\omega^{p^n}=\omega$, we have $\triangle=X^{p^n+1}-Y^{p^n+1}$.

We are to determine the number of elements X, Y of the $GF[p^{2n}]$ for which $\triangle=0$. Then $X=RY$ where R is one of the p^n+1 elements for which $R^{p^n+1}=1$. According as Y does not equal 0 or $Y=0$, we obtain $(p^n+1)(p^{2n}-1)$ sets or 1 set X, Y. By (4) each set determines uniquely a set A, B, C, D in the $GF[p^n]$. Hence there are

*This result may also be obtained by subtracting from $(p^n)^4$ the number of sets for which $xy-zw$ does not equal 0, which is the order $(p^{2n}-1)(p^{2n}-p^n)$ of the general binary linear group in the $GF[p^n]$.

$$(p^n+1)(p^{2n}-1)+1=p^n(p^{2n}+p^n-1)$$

distinct sets A, B, C, D for which $\triangle=0$.

Whether ω belongs to the $GF[p^n]$ or does not, the number of sets A, B, C, D making $\triangle=0$ is therefore $p^n(p^{2n}+p^n-1)$. It remains to determine for each set the number of elements I, γ for which f is not equal to 0, f_1 is not equal to 0. For p not equal to 2, this number is $(p^n-1)^2$ since $\mathrm{I}+\gamma$ and $\mathrm{I}-\gamma$ must take independently p^n-1 values. For $p=2$, $f=f_1$ and the number is $(2^n-1)2^n$.

The total number of sets I, α, ..., ε for which $D=0$ is therefore

$p^n(p^{2n}+p^n-1)(p^n-1)^2+p^{2n}(p^{3n}-p^{2n})+p^{5n}$ (if p is not equal to 2, p not equal to 3.)

$2^n(2^{2n}+2^n-1)(2^n-1)2^n+2^{5n}$ (if $p=2$).

Subtracting these numbers from p^{6n} and 2^{6n} respectively, we obtain the order of G. The results may be combined into the theorem:

The order of the group G in the $GF[p^n]$ is $p^n(p^n-1)^4(p^n+1)$ if $p>3$; $3^{4n}(3^n-1)^2$ if $p=3$; $2^{2n}(2^n-1)^3(2^n+1)$ if $p=2$.

Consider the group H the matrix of whose general transformation is derived from the matrix (1) by setting $\alpha=\beta$, $\gamma=\delta=\varepsilon$, the equal elements corresponding to the conjugate operators in the symmetric group g_6. The determinant D of this special matrix is the special group-determinant of g_6. Then

$$(5) \qquad D'=(\mathrm{I}+2\alpha+3\gamma)(\mathrm{I}+2\alpha-3\gamma)(\mathrm{I}-\alpha)^4.$$

For $p=3$, $D'\equiv(\mathrm{I}-\alpha)^6$, so that the order of H is $3^{2n}(3^n-1)$.

For $p=2$, $D'\equiv(\mathrm{I}-\gamma)^2(\mathrm{I}-\alpha)^4$, so that the order of H is $2^n(2^n-1)^2$.

For $p>3$, we determine the number of sets I, α, γ in the $GF[p^n]$ for which $D'=0$. These are of three classes:

f' not equal to 0, f_1' not equal to 0, $\mathrm{I}-\alpha=0$;
f' not equal to 0, $f_1'=0$; $f'=0$,

where $f'=\mathrm{I}+2\alpha+3\gamma$, $f_1'=\mathrm{I}+2\alpha-3\gamma$. For the first class, $\mathrm{I}=\alpha$, $3\alpha\pm3\gamma$ is not equal to 0.

If $\alpha=0$, γ has p^n-1 values; if α is not equal to 0, γ has p^n-2 values. Hence

$$(p^n-1)+(p^n-1)(p^n-2)=(p^n-1)^2$$

is the number of sets in the first class. For the second class,

$$\mathrm{I}+2\alpha-3\gamma=0,\ \gamma \text{ is not equal to } 0,$$

giving $(p^n-1)p^n$ sets. The third class contains p^{2n} sets. Hence

$$(p^n-1)^2+(p^n-1)p^n+p^{2n}=3p^{2n}-3p^n+1$$

is the total number of sets making $D'=0$. Subtracting this number from p^{3n}, we obtain $(p^n-1)^3$ as the order of H for $p>3$.

The group* H is an invariant subgroup of G. In view of the preceding results, the order of the quotient-group is

$$p^n\,(p^{2n}-1) \text{ if } p>3 \text{ or if } p=2\,;\; 3^{2n}(3^n-1) \text{ if } p=3.$$

This result for p not equal to 3 is in accord with the general theory of group-matrices† by which the quotient-group is seen to be simply isomorphic with the group of binary substitutions of determinant unity in the $GF[p^n]$. The latter is known to be simple if $p=2$, and to have the factors of composition $\frac{1}{2}p^n\,(p^{2n}-1)$ and 2 if $p>2$.

The University of Chicago, March, 1902.

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*The group H is evidently simply isomorphic with the commutative group of ternary linear transformations whose general matrix is

$$\begin{Bmatrix} 1 & 2\alpha & 3\gamma \\ \alpha & 1+\alpha & 3\gamma \\ \gamma & 2\gamma & 1+2\alpha \end{Bmatrix}$$

†Frobenius, Burnside, Dickson. See the references in the *Transactions of the American Mathematical Society*, July, 1902.

"THE BETWEENNESS ASSUMPTIONS."

By DR. ELIAKIM HASTINGS MOORE.

Amongst mathematicians there is abiding interest in the foundations of geometry—at present, in particular, as to the projective axioms. These axioms constitute, for instance, the first two groups I, II of Hilbert's system of axioms.

In a paper "On the Projective Axioms of Geometry," published January, 1902, in The Transactions of the American Mathematical Society (vol. 3, pp. 142-158), I exhibited and developed a new system of projective axioms for geometry of three or more dimensions, comparing it with the systems of Pasch, of Peano, and of Hilbert, and in this connection proving the redundancy in Hilbert's system I, II of the axioms I 4, II 4.

In the April 1902 number of THE MONTHLY (pp. 98-101), under the title, "The Betweenness Assumptions," Dr. Halsted published a second proof of the redundancy of II 4, a proof due to Mr. R. L. Moore, a student of his. Dr. Halsted alluded to my earlier proof of the theorem in the statement: "Mr. Moore has no intimation that any one has ever tried to prove these theorems." (l. c. p. 100.)

I wrote to Mr. Moore explaining the situation, and congratulating him upon the beauty of his proof. The congratulatory part of this letter appears in an editorial note (p. 148) of the May 1902 number of THE MONTHLY.

The letter was as follows:

THE UNIVERSITY OF CHICAGO, May 6, 1902.

MR. R. L. MOORE, The University of Texas, Austin, Texas.

MY DEAR MR. MOORE: I read with much interest, the other day, your proof of the redundancy of Hilbert's axiom II 4, in his system I, II, as exhibited by Professor Halsted in the current number of the AMERICAN MATHEMATICAL MONTHLY. Today I received from Professor Halsted a copy of that number. This is in response to a letter I sent him a week or so ago stating that I should be pleased to receive for publication in the Transactions the delightfully simple proof of the redundancy of which he wrote [had written] to me. I certainly agree with him in this estimate of your proof. Apparently he has not called your attention to the fact that the redundancy was pointed out by me and proved in my paper, which I am sending under separate cover, on the projective axioms of geometry, published in the January number of the Transactions. In accordance with correspondence with him, it was in connection with this paper of mine that he wrote to Hilbert and received Hilbert's response which led to your work on the subject. You will see that it was my desire to survey the whole system of projective axioms, and to exhibit a new system, and, in that connection to show that Hilbert's axioms I 4 and II 4 were in his system redundant, and, moreover, to furnish a satisfactory account of the rôles of the axioms I 3, 4, 5 which had been held by Schur to be redundant. As to the axiom II 4, you will see that, by considerations of the other linear axioms alone, and so in particular without the use of II 5, or of my axiom 4, I prove on page 151 that the axiom II 4 is a result of the statement 2_1, which [statement] is the statement of your theorem I. Thus to complete the proof of the redundancy of II 4, in Hilbert's system, I should today make use of your proof of theorem I. The proof that I give, in that it involves my triangle transversal axiom 4, is necessarily much longer.

I have supposed that you might be interested in understanding how your paper impresses me, and remain with considerable interest in the progress of your mathematical career,

Yours very truly,

(Signed) E. H. MOORE.

I suppose that the letter as a whole may be of value and interest to some readers of THE MONTHLY.

Chicago, June 3, 1902.

A NON-EUCLIDEAN GEM.

By DR. GEORGE BRUCE HALSTED.

La Géométrie non-euclidienne. Par P. Barbarin. Paris, C. Naud. 1902.

It is peculiarly appropriate that from Bordeaux, made sacred for non-Euclidean geometry by Hoüel, should emanate this beautiful little treatise, decor-

ated with a 'gravure' reproducing part of a manuscript of Euclid, also with the 'official' portrait of Lobachevski, but best of all with a portrait of Riemann.

It begins from the hackneyed position: "Experience therefore it is which has furnished to the ancient geometers a certrin number of primitive notions, of axioms, or fundamental postulates put by them at the basis of the science."

But now we know there never was any pure receptivity. In all thinking enters a creative element. Every bit of experience is in part created by the subject said to receive it, but really in great part making it.

Professor Barbarin continues: "From the epoch of Euclid, this number has been reduced to the strict minimum necessary, and all the others not comprised in this list, being capable of demonstration, are put in the class of theorems." Now we know that Euclid omits to notice many of the assumptions he unconsciously employs, for example all the "betweenness assumptions," while Hilbert has at last rigorously demonstrated Euclid's assumption, "All right angles are equal," and in turn one of Hilbert's assumptions has just been proven (see AMERICAN MATHEMATICAL MONTHLY, April, 1902, pp. 98-101).

The 'Elements' of Euclid, says Professor Barbarin, enjoyed throughout all the middle ages and still enjoy a celebrity that no other work of science has attained; this celebrity is due to their logical perfection, to the admirable concatenation of the propositions, and to the rigor of the demonstrations.

"Il mit dans son livre," says Montucla, "cet enchaînement si admiré par les amateurs de la rigueur géométrique." "In vain," he adds, "divers geometers whom this arrangement has displeased, have attempted to better it. Their vain efforts have made clear how difficult it is to substitute for the chain made by the Greek geometer another as firm and as solid." 'This opinion of the historian of mathematics,' says our author, 'retains all its value even after the researches which geometers have undertaken for about a century to submit the fundamental principles of the science to an acute and profound examination.'

I add that the remarkable discoveries of Dehn (see *Science*, N. S., Vol. XIV, pp. 711, 712), prove an unexpected superiority for Euclid over all successors down to our very day, and suggest the latest advance, which, though as yet unpublished, exists, for under date of April 2, 1902, Hilbert writes me: "In einer andern Arbeit will ich die Lobatschefski'sche Geometrie in der ebene unabhangig von Archimedes begrunden.' That is, Hilbert will found Bolyai's geometry as he has Euclid's, without any continuity assumption. To get the benefit of this brilliant achievement, I am holding back my own book on this fascinating subject.

Says Hilbert in his unpublished Vorlesung ueber Euklidische Geometrie, "The order of propositions is important. Mine differs strongly from that usual in text books of elementary geometry; on the other hand, it greatly agrees with Euclid's order.

"So fuehren uns diese ganz modernen Untersuchungen dazu, den Scharfsinn dieses alten Mathematikers recht zu wuerdigen und aufs hoechst zu bewundern."

Again, *apropos* of Euclid's renowned parallel postulate, Hilbert says: "What sagacity, what penetration the setting up of this axiom required we best recognize if we look at the history of the axiom of parallels. As to Euclid himself (circa 300 B. C.) he, *e. g.*, proves the theorem of the exterior angle before introducing the parallel axiom, a sign how deeply he had penetrated *in den Zusammenhang der geometrischen Saetze.*"

Professor Barbarin repeats the exploded error of attributing to Gauss the discovery of the non-Euclidean geometry in 1792. In the introduction to my translation of Bolyai's 'Science Absolute of Space,' pp. viii-ix, is a letter from Gauss, on which I there remark: "From this letter we clearly see that in 1799 Gauss was still trying to prove that Euclid's is the only non-contradictory system of geometry, and that it is the system regnant in the external space of our physical experience. The first is false; the second can never be proven." In 1804 Gauss writes that in vain he still seeks the unloosing of this Gordian knot.

Again, with the date April 27, 1813, we read: "In the theory of parallels we are even now not farther than Euclid was. This is the *'partie hortense'* (shameful part) of mathematics, which soon or late must receive a wholly different form. Thus in 1813 there is in Göttingen still no light.

But in 1812 in Charkow, the non-Euclidean geometry already had been for the first time consciously created by Schweikart, whose summary characterization of it is given in *Science*, N. S., Vol. XII, pp. 842-846. This he communicated to Bessel and sent to Gerling and afterward to Gauss in 1818, so that it may claim to be the first *published* (not printed) treatise on non-Euclidean geometry. By this time Gauss had progressed far enough to be willing to signify *privately* his acceptance of Schweikart's doctrines.

On p. 15 Barbarin makes a brief argument for Euclid's axiom, "All right angles are equal." This argument was good before Hilbert and Veronese, since this axiom can never be proved by superposition. It is already a consequence of the assumptions preliminary to motion. This profounder analysis Barbarin has not attained to. He still uses as a postulate and supposes indispensable "l indeformabilité des figures en déplacement." What Tules Andrade calls 'cette malheureuse et illogique définition' of Legendre, 'the shortest path between two points is a straight line,' Barbarin puts as an elementary proposition! Manning also, p. 2, assumes it, thus invalidating and making ephemeral his pretty little "Non-Euclidean Geometry" (Ginn & Co., 1901). Barbarin then proceeds to classify geometries by Saccheri's three hypotheses, the hypothesis of obtuse angle, the hypothesis of right angle, the hppothesis of acute angle, or that the angle sum of a rectilineal triangle is greater than, equal to, less than two right angles. But the remarkable discoveries of Dehn have now shown that this classification is invalid.

Barbarin says, p. 16, 'Saccheri proves that the hypothesis of the obtuse angle is incompatible with postulate 6' of Euclid. Dehn dissipates this supposed incompatibility by actually exhibiting a new geometry in which they amicably blend, which he calls the non-Legendrean geometry. In the same way,

the hypothesis of right angle amalgamates with the contradiction of Euclid's parallel-postulate in a geometry which Dehn calls semi-Euclidean. As Dehn states this result: There are non-Archimedian geometries in which the parallel-axiom is not valid and yet the angle-sum in every triangle is equal to two right angles. Thus the Theorem (Legendre, 12th Ed., I, 23; Barbarin, p. 25): 'If the sum of the angles of every triangle is equal to two right angles, the fifth postulate is true,' is seen to break down.

Manning's 'Non-Euclidean Geometry,' though it says (p. 93), 'The elliptic geometry was left to be discovered by Riemann,' gives only the single elliptic. It never even mentions the double elliptic, or spherical or Riemannian geometry, which Killing maintains was the only form which ever came before Riemann's mind. If so, then Barbarin's book is like Riemann's mind. The Riemannian, as distinguished from the single elliptic, is the only form which appears in it. Killing was the first, who (1879, *Crelle's Journal*, Bd. 83) made clear the difference between the Riemannian and the single elliptic space (or as he calls it, the polar form of the Riemannian).

Klein championed the single elliptic. Manning knows no other. Professor Simon Newcomb. like Manning, deals only with the single elliptic in his treatise; 'Elementary theorems relating to the geometry of a space of three dimensions and of uniform positive curvature *in the fourth dimension*.' The last four words F. S. Woods replaces by seven dots in his article 'Space of constant curvature' (*Annals of Math.*, Vol. 3, p. 72), though blaming Professor E. S. Crawley for the error they contain. Newcomb's also was the unfortunate statement which dubbed this "A Fairytale of Geometry,' a point of view from which he is still suffering in his latest little unburdening in *Harper's Magazine*. Just so Lobachevski had the misfortune to call his creation "Imaginary Geometry." Contrast John Bolyai's "The Science Absolute of Space."

In single elliptic space every complete straight line is of finite constant length πk. Every pair of straight lines intersect and return again to their point of intersection, but have no other point in common. In the so-called spherical space, that is the Riemannian space, two straight lines always meet in two points (opposites, or antipodal points,) which are πk from each other. The single elliptic makes the plane a unilateral or double surface, so that two antipodal points would correspond to one point, but to opposite sides of this one-sided plane with reference to surrounding three-dimensional elliptie space.

The geometry for two-dimensional Riemannian space coincides completely with pure spherics, that is, with spherics established from postulates which make no reference to anything off of the sphere, inside or outside the sphere. Hence the great desirability of a treatise on pure spherics. It would at the same time be true and available for Euclidean and for Riemannian geometry. Yet its relations to three-dimensional Euclidean and three-dimensional Riemannian space would differ radically.

Through every Riemannian straight line passes an infinity of planes also Riemannian, and in each of these this straight has a determined and distinct

center; but the straight is independent of the planes, and is defined by the postulates. Now in the sphere the great circle and the one *pseudo*-plane which contains and fixes it, namely the sphere, are inseparable, since any portion, however minute, of either determines all the other as well as its center and radius.

In the single elliptic geometry the elliptic straight line does not divide the elliptic plane into two separated regions. We can pass from any one point of the plane to any other point without crossing a given straight in it. Starting from the point of intersection of two straights and passing along one of them a certain finite length, we come to the intersection point again without having crossed the other straight. Hence we can pass from what seems one side of the straight line to what seems the other without crossing it, that is, it is uni-lateral or double.

This single elliptic geometry is never mentioned in Barbarin's book; just as the Riemannian is never mentioned in Manning's book. First take your choice, then buy your non-Euclidean geometry.

On p. 36, Barbarin gives to Gauss the honor which belongs to Wallis of being the first to remark that the existence of unequal similar figures is equivalent, in continuous space, to the parallel postulate.

In Chapter VII, 'Les Contradicteurs de la géométrie non Euclidienne,' Professor Barbarin makes with unanswerable vigor the argument which I gave in my 'Report on Progress in Non-Euclidean Geometry,' (*Science*, N. S., Vol. X, pp. 545-557). There I quoted Whitehead, who was the first to publish (March 10, 1898) "the extension of Bolyai's theorem by investigating the properties of the general class of surfaces in any non-Euclidean space, elliptic or hyperbolic, which are such that their geodesic geometry is that of straight lines in a Euclidean plane. "Such surfaces are proved to be real in elliptic as well as in hyperbolic space, and their general equations are found for the case when they are surfaces of revolution. In hyperbolic space, Bolyai's limit-surfaces are shown to be a particular case of such surfaces of revolution. The same principles would enable the problem to be solved of the discovery in any kind of space of surfaces with their 'geodesic' geometry identical with that of planes in any other kind of space."

Now not only the strikingly important problem solved by Whitehead, but also the analagous problem indicated, had both been solved by Barbarin and presented three months before to the Académie Royale de Belgique; but these investigations were only published after the appearance of my Report (October 20, 1899). They, as Barbarin says, p. 63, 'bring out in a striking manner the absolute independence of the three systems of geometry, which are able each to get everything from its own resources without need of borrowing anything from the others.' In each of the three spaces, Euclidean, Bolyaian, Riemannian, there exist surfaces whose geodesics have the metric properties of the straights of the two other spaces.

But the book in which these beautiful researches are published: 'Etudes de géometrie analytique non euclidienne par P. Barbarin, Bruxelles,' 1900, Hayez, pp. 168, has other titles to universal recognition.

Notwithstanding the ever-present example of Euclid, who never uses a construction or a figure which he has not introduced as following deductively from his two postulated figures, the straight and the circle, an insidious error crept into geometry, taught by Beman and Smith in the following words: (See their 'Geometry,' 1899, p. 70, § 112, or the AMERICAN MATHEMATICAL MONTHLY, Vol. IX, p. 131.)

"*Note on Assumed Constructions.* It has been assumed, up to prop. XXVIII, that all constructions were made as required for the theorems. Thus an equilateral triangle has been frequently mentioned, although the method of constructing one has not yet been indicated; a regular heptagon has been mentioned in ex. 93, and reference might be made to certain results following from the trisection of an angle, although the solutions of the problems, to construct a regular heptagon, and to trisect any angle, are impossible by elementary geometry. But the possibility of solving such problems has nothing to do with the logical sequence of the theorems."

This is a fundamental error. Thus for example, to construct an equilateral triangle on a given sect ['line segment,' they say,] requires and presumes the following lemma: "If A and B be any two given points, there is at least one point C whose sects from A and B are both congruent to the sect AB." This requires for its deduction a continuity assumption, such as that of Dedekind, Weierstrass, or Archimedes. But this totally changes the whole sequence of theorems. Hilbert establishes Euclidean geometry without any such assumption. His assumption of this construction would shatter his whole edifice. Moreover the fact that, even with a continuity assumption, in elliptic space it is not always, even then, possible to construct an equilateral triangle on a given base, shows that the assumption involves also the previous assumption that the straight line is not finite or closed, which again involves the betweenness assumptions so brilliantly reduced from five to four in the April number of the AMERICAN MATHEMATICAL MONTHLY, again involving the whole logical sequence of the theorems.

The construction so glibly assumed, to pass a circle through any three non-co-straight points, is equivalent to the assumption of the world-renowned parallel-postulate, and thus has everything in the world to do with the sequence of the theorems. The assumed construction of a triangle from three sects which are to be its sides, by the method of Beman and Smith, p. 76, is equivalent to the assumption of the Archimedes postulate, which again has everything to do with the logical sequence of the theorems. In fact just this assumption makes ephemeral the beautiful method of Saccheri used in the book we are reviewing.

Hence we can appreciate that astounding achievement of Bolyai's young genius, his § 34, where he solves for his universe, Eu., I, 31, To draw a straight line through a given point parallel to a given straight line. His brilliant lead was followed more than half a century later by Gerard, but it is Barbarin who has ended the matter by deducing from certain very simple constructions of the trirectangular quadrilateral all the fundamental plane constructions.

In chapter VIII ('La géométrie physique,' § 30, 'La forme géométrique de notre univers') our author stresses the idea, that even if our universe were exactly Euclidean, it would be forever impossible for us to demonstrate this. As I said in my 'Non-Euclidean Geometry for Teachers,' p. 14: "If in the mechanics of the world independent of man we were absolutely certain that all therein is Euclidean and only Euclidean, then Darwinism would be disproved by the reductio ad absurdum. All our measurements are finite and approximate only. The mechanics of actual bodies in what Cayley called the external space of our experience, might conceivably be shown by merely approximate measurements to be non-Euclidean, just as a body might be shown to weigh more than two grams or less than two grams, though it never could be shown to weigh precisely, absolutely two grams."

Our author suggests the following experiment for proving our space non-Euclidean: From a point trace six rays sixty degrees apart. On them successively mark off the sects OA_0, OA , OA_2, ..., OA_n, of which each is the projection of the following. If we finish by finding between OA_n and 2^nOA_0 a difference of constant sense and greater than imputable to error of procedure, our universe is non-Euclidean.

In conclusion this beautiful little book has the advantage of being the production of an active and fertile original worker in the domain of which it treats. His 'Géométrie general des espaces' (1898), his 'Sur le paramétre de l' univers, and 'Sur la géométrie des êtres plans' (1901), 'Le cinquiéme livre de la métagéométrie,' (1901), 'Les cosegments et les volumes en géométrie non euclidienne' (1902), and his 'Poligones réguliers spheriques et non-euclidienns,' shortly to appear in that virile young monthly, *Le Matematiche*, and which I had the advantage of reading in manuscript, show that Bordeaux is honored by a worthy successor of Hoüel, so universally beloved.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

Remark on Problem No. 154. by G. B. M. Zerr.

According to the correction made in the April number, the result in the second solution is 17. This result is not correct. The problem as stated is not possible. It takes 12 oxen to eat the growing grass. Then $17-12=5$ oxen remaining to eat the grass already grown.

Now 9 oxen eat the standing grass in 6 weeks or 1 ox eats it in 54 weeks.

$\therefore$ 5 oxen will eat it in $54\div5=10\frac{4}{5}$ weeks.

$\therefore$ 17 oxen will eat it, the grass, together with what grows, in $10\frac{4}{5}$ weeks.

157. Proposed by B.F.FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

January 1, 1899, A and B entered into partnership for 3 years. A put in $10,000 and B put in $5,500. July 1, 1899, B put in $1,500 more. October 1, A took out $500. January 1, 1900, each put in $1,500. July 1, 1900, they dissolved partnership, and found that they had lost $846. What is each partner's share of the loss?

Solution by J. R. HITT, Choral Institute, San Marcos, Texas, and HON. JOSIAH H. DRUMMOND, Portland, Maine.

A has in $10,000 for 9 months, $9,500 for 3 months, $11,000 for 6 months.

B has in $5,500 for 6 months, $7,000 for 6 months, $8,500 for 6 months.

Assuming the loss to be 10% of investment, A's loss would be $750+$237.50+$550=$1537.50. B's would be $275+$350+$425=$1050.

Hence, $1537.50+$1050 : $846=$1537.50 : $502.696, A's loss.

$1537.50+$1050 : $846=$1050 : $343.304, B's loss.

Also solved by *G. B. M. ZERR*. Professor Hitt should have received credit for solving 156.

158. Proposed by JAMES F. LAWRENCE, A. B., Professor of Mathematics, Rogers Academy, Rogers, Ark.

My agent sold pork at 5% commission; increasing the proceeds by $20, I ordered the purchase of flour at 3% commission; after which flour rose 9%, my whole gain was $40. What did he sell the pork for?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let 100%=selling price of pork.

100% + $20=total cost.

$(95\% + \$20)\frac{109}{103} = 100\frac{55}{103}\% + \$21\frac{17}{103}$, selling price of flour.

$100\frac{55}{103}\% + \$21\frac{17}{103} - 100\% - \$20 = \$40$ gain.

$\therefore\ \frac{55}{103}\% = \$38\frac{86}{103}$. $\therefore\ 1\% = \$72\frac{8}{11}$.

$100\% = \$7272\frac{8}{11}$, selling price of pork.

Also solved by *J. R. HITT*.

ALGEBRA.

145. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

Factorize $2b^2c^2+2c^2a^2+2a^2b^2+2a^2d^2+2b^2d^2+2c^2d^2-a^4-b^4-c^4-d^4$.

No correct solution of this problem has been received.

146. Proposed by B. F. YANNEY, Professor of Mathematics, Mount Union College, Alliance, Ohio.

If the series 1, 3, 5, ... $2n-1$, ... be divided into successive groups of r terms each, the sum of the terms of the nth group will be $(2n-1)$ times the sum of the terms of the first group, or $(2n-1)r^2$.

Solved by H. S. VANDIVER, Bala, Pa., and E. D. GRABER, Professor of Mathematics, Geneseo State Normal School, Geneseo, N. Y.

The n groups in question are

$$\begin{array}{l} 1,\ 3,\ 5,\ \ldots.\ 2r-1 \\ 2r+1,\ 2r+3,\ \ldots.\ 4r-1 \\ 4r+1,\ 4r+3,\ \ldots.\ 6r-1 \\ \vdots \qquad \vdots \qquad \vdots \\ 2(n-1)r+1,\ \ldots.\ 2nr-1 \end{array}$$

The sum of the terms in the nth group is

$$\frac{r}{2}(\text{1st term}+\text{last term})=\frac{r}{2}(4rn-2r)=r^2(2n-1).$$

Solved similarly by *G. B. M. ZERR*, *J. H. DRUMMOND*, and *J. SCHEFFER*.

147. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, Gloucestershire. England.

Prove that $x=a^x$ has never more than two real roots, and find the condition for no real roots.

No solution of this problem has been received.

148. Proposed by R. D. BOHANNAN, Ph. D., Professor of Mathematics, Ohio State University, Columbus, O.

If $\frac{x}{a+\alpha}+\frac{y}{b+\beta}+\frac{z}{c+\gamma}=1$, $\frac{x}{a+\beta}+\frac{y}{b+\beta}+\frac{z}{c+\beta}=1$, $\frac{x}{a+\gamma}+\frac{y}{b+\gamma}+\frac{z}{c+\gamma}=1$, show, without solving, that $x+y+z=a+\alpha+b+\beta+c+\gamma$.

No solution of this problem has been received.

149. Proposed by JOSEPH V. COLLINS, Ph. D., Stevens Point, Wis.

1. How many different football elevens can be sent out from a school having twenty players? In how many ways can eleven men line up?

Solution by P. H. PHILBRICK, C. E., Lake Charles, La.

It is possible to send out $\frac{20!}{11!\ 9!}$ elevens, or 167960 elevens.

The eleven men can line up 11! ways.

Also solved by *G. B. M. ZERR* and *C. A. LINDEMANN*.

150. Proposed by JOSEPH V. COLLINS, Ph. D., Stevens Point, Wis.

2. How many sets of officers (president, vice-president, treasurer, and secretary) can a society of forty persons elect? How many committees of four persons, supposing no attention is paid to positions on the committees? How many committees in which the chairman is selected?

Solution by P. H. PHILBRICK, C. E., Lake Charles, La., and C. A. LINDEMAN, Professor of Mathematics Virginia Union University, Richmond, Va.

The society can elect $\frac{40!}{36!.4!}$ sets of officers.

The number of committees, no attention being paid to positions on the

same, is also $\dfrac{40!}{36!.4!}=\dfrac{40.39.38.37}{1.2.3.4}$.

The number of committees in which the chairman is selected, leaving 39 from whom to choose, is $\dfrac{39!}{36!.3!}=\dfrac{39.38.37}{6}=9139$.

Also solved by *G. B. M. ZERR.*

151. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Solve the equations:

$$\begin{aligned}&x+y+z+u+w=1.\\&ax+by+cz+du+ew=h,\\&a^2x+b^2y+c^2z+d^2u+e^2w=h^2,\\&a^3x+b^3y+c^3z+d^3u+e^3w=h^3,\\&a^4x+b^4y+c^4z+d^4u+e^4w=h^4.\end{aligned}$$

Solution by CLARENCE E. COMSTOCK, Professor of Mathematics. Bradley Polytechnic Institute. Peoria, Ill.

Solution by determinants.

$$x=\begin{vmatrix}1&1&1&1&1\\h&b&c&d&e\\h^2&b^2&c^2&d^2&e^2\\h^3&b^3&c^3&d^3&e^3\\h^4&b^4&c^4&d^4&e^4\end{vmatrix}\div\begin{vmatrix}1&1&1&1&1\\a&b&c&d&e\\a^2&b^2&c^2&d^2&e^2\\a^3&b^3&c^3&d^3&e^3\\a^4&b^4&c^4&d^4&e^4\end{vmatrix}\equiv\frac{\Delta_{a=b}}{\Delta}.$$

By the factor theorem, we get

$\Delta=(a-b)(a-c)(a-d)(a-e)(b-c)(b-d)(b-e)(c-d)(c-e)(d-e)$.

$\Delta_{a=h}$=the same with a replaced by h.

$$\therefore x=\frac{(h-b)(h-c)(h-d)(h-e)}{(a-b)(a-c)(a-d)(a-e)}.$$

Since a, b, c, d, e appear in the same way, the principle of symmetry enables us to write the values y, z, u, w at once.

$$y=\frac{(h-a)(h-c)(h-d)(h-e)}{(b-a)(b-c)(b-d)(b-e)},\ z=\frac{(a-h)(c-h)(d-h)(e-h)}{(b-a)(c-a)(d-a)(e-a)},$$

$$u=\frac{(a-h)(b-h)(c-h)(e-h)}{(a-d)(b-d)(c-d)(e-d)},\ \text{and}\ w=\frac{(a-h)(b-h)(c-h)(d-h)}{(a-e)(b-e)(c-e)(d-e)}$$

Solved in a similar manner by *G. B. M. ZERR.*

152. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Solve by a short original method, if possible:

$$x/a+y/b+c/z=P\ldots.(1),$$
$$x/a+b/y+z/z=Q\ldots.(2),$$
$$a/x+y/b+z/z=R\ldots.(3).$$

Solution by the PROPOSER.

Let $x/a=u$, $y/b=v$, $z/c=w$.

$u+v+1/w=P\ldots.(1)$, $u+1/v+w=Q\ldots.(2)$, $1/u+v+w=R\ldots.(3)$.

$(1)-(2)$ gives $v^2\mathrm{w}+v-vw^2-Pvw+Qvw=w\ldots.(4)$.

$(1)\times(3)$ gives $1=(P-v-1/w)(R-v-w)$, or

$$v^2w+v+vw^2-Pvw-Rvw=Pw^2-PRw+R\ldots.(5).$$

$(5)-(4)$ gives $vw=\dfrac{(w-R)(Pw-1)}{2w-R-Q}\ldots.(6)$.

$(5)\div(4)$ gives $\dfrac{vw+1+w^2-Pw-Rw}{vw+1-w^2-Pw+Qw}=\dfrac{Pw^2-PRw+R}{w}\ldots.(7)$.

(6) in (7) gives

$$\frac{2w^3-(P+Q+3R)w^2+(1+R^2+PQ+RQ)w-Q}{-2w^3+(R-P+3Q)w^2+(1-Q^2+PQ-RQ)w-Q}=\frac{Pw^2-PRw+R}{w}$$

$\therefore$ $2Pw^5+(2-3PR-3PQ+P^2)w^4+(4PQR+PR^2-P^2R+PQ^2-P^2Q-2P-Q-R)w^3+(1+2PQ-2QR+2PR-PQ^2R+P^2QR-PR^2Q)w^2+(RQ^2+R^2Q-R-Q-2PQR)w+RQ=0$.

This is an equation of the 5th degree and I have thus far been unable to solve it.

Let $P=Q=R$. Then

$$2Pw^5+(2-5P^2)w^4+4P(P^2-1)w^3+(1+2P^2-P^4)w^2-2Pw+P^2=0.$$

Let $P=3$.

$\therefore$ $w^5-\frac{43}{6}w^4+16w^3-\frac{31}{3}w^2-w+\frac{3}{2}=0$, or $(w-\frac{1}{2})(w-1)(\mathrm{w}-3)(\mathrm{w}-3)(\mathrm{w}-\frac{1}{3})=0$.

$\therefore$ $\mathrm{w}=\frac{1}{2}$, 1, 3, 3 or $-\frac{1}{3}$; $v=\frac{1}{2}$, 1, 3, $-\frac{1}{3}$ or 3; $u=\frac{1}{2}$, 1, $-\frac{1}{3}$, 3 or 3.

GEOMETRY.

176. Proposed by R. A. WELLS, Franklin College, New Athens, Ohio.

If there be three straight lines which meet in a point, and the arbitrary constants of their equations, expressed in the slope form, be taken as the co-ordinates of three points, these three points will lie in a straight line.

Solution by ANNA L. VAN BEUSCHOTEN, Professor of Mathematics, Wells College, Aurora, N. Y.

Let the three straight lines be given by the equation

$$y=ax+b,\ y=cx+d,\ y=ex+f.$$

The condition that these lines intersect in a common point is given by the vanishing of the determinant,

$$\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$$

But the vanishing of this determinant is also the condition that the points (a, b), (c, d), (e, f) are colinear.

Also solved by *G. B. M. ZERR*.

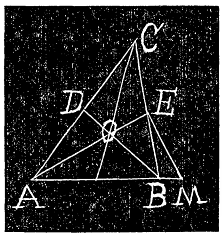
177. Proposed by GEORGE LILLEY, Ph. D., LL. D., University of Oregon. Eugene. Ore.

If two medians of a triangle intersect each other at right angles, the third median will be the hypotenuse of a right triangle, of which the other two will be the sides.

Solution by H. B. PENHOLLOW. DeWitt Clinton High School, New York. N. Y.

Given $\triangle ABC$, medians meeting at O, having $\angle AOB$ a right angle.

From E draw EM perpendicular to AE, meeting AB produced in M. Then $\triangle AEM$ is a right triangle in which AE is one median, $EM=DB$ another median. Also since triangles AEM and AOB are similar, $AE/AO=AM/AB$. But $AE=\frac{3}{2}AO$.



$\therefore AM=\frac{3}{2}AB$.

Also OF is median of right triangle AOB.

$\therefore OF=\frac{1}{2}AB$, or $CF=\frac{3}{2}AB=AM$. Q. E. D.

Also solved by *P. S. BERG, HENRY HEATON, P. H. PHILBRICK, C. A. LINDEMANN, G. I. HOPKINS, S. E. HARWOOD, J. F. LAWRENCE, T. T. DAVIS, G. B. M. ZERR*, and *ANNA BENCHOTEN*. Professor Penhollow and Miss Benchoten each furnished three solutions.

178. Proposed by JOHN M. ARNOLD. Crompton. R. I.

A cylinder thirty feet long and two feet in diameter is to be placed in a machinery car, the inside dimensions of which are eight feet wide and eight feet high. Find length of the shortest car that will contain it.

No correct solution of this problem has been received.

179. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University of Mississippi.

Of all isosceles triangles inscribed in a circle, the equilateral is the maximum and has the maximum perimeter. Prove geometrically.

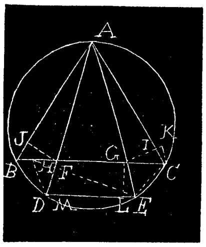
Solution by the PROPOSER.

Case I. (See Fig. 1.) Vertical angle of isosceles triangle less than 60°.

Let ABC be an inscribed equilateral triangle and ADE any inscribed isosceles triangle with its base DE parallel to BC.

1st. To prove $\triangle ABC$ greater than $\triangle ADE$. $\triangle AFG$ is common.

Through D draw a parallel to EA meeting BC at H, and forming the parallelogram $DHGE$. It is easily proved that H always lies to the right of B.



Hence, HG is less than BG. Draw the diagonal HE.

Comparing triangles HGE and AGC,

$$\triangle HGE : \triangle AGC = HG.GE : AG.GC.$$

In $\triangle BAG$, BG is less than AG, and therefore, HG is less than AG. Also, in $\triangle GEC$, GE is less than GC.

Therefore, $\triangle HGE$ is less than $\triangle AGC$.

Hence parallelogram $HGED$ is less than $\triangle AGC + \triangle AFB$, and trapezoid $DFGE$, being a part of this parallelogram, is less than the sum of the same two triangles.

Therefore $\triangle ABC$ is greater than $\triangle ADE$.

2nd. To prove the perimeter of $\triangle ABC$ greater than that of $\triangle ADE$.

Draw FJ and GI perpendicular to AF and AG, respectively.

Draw CK perpendicular to GI produced, and FM and GL perpendicular to DE. AI is greater than AG, AJ is greater than AF, $FG = ML$.

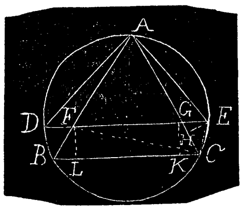
In similar triangles CGK and GLE, CG is greater than EG, CK (and, therefore, CI) is greater than EL.

Similarly, BF is greater than DF, and BJ is greater than DM.

It follows that the perimeter of $\triangle ABC$ is greater than that of $\triangle ADE$.

Case II. (See Fig. 2.) Vertical angle of isosceles triangle greater than 60°.

Let ABC be an inscribed equilateral triangle and ADE an inscribed isosceles triangle with its base, DE, parallel to BC.



1st. To prove $\triangle ABC$ greater than $\triangle ADE$.

$\triangle AFG$ is common. Draw FC.

$\triangle FGC$ is greater than $\triangle AGE$, since $FG = AG$, and in $\triangle GCE$, GC is greater than GE.

It follows that trapezoid $BFGC$ is greater than the sum of triangles AGE and AFD.

Hence $\triangle ABC$ is greater than $\triangle ADE$.

2nd. To prove perimeter of $\triangle ABC$ greater than that of $\triangle ADE$.

On AC lay off AH equal to AE. Connect H and E.

$\angle AEH = \angle AHE$. $\angle GEH = \angle AEH - \angle AEG$.

Also, $\angle CEH = \angle AHE - \angle HCE$. But $\angle HCE = \angle AEG$.

Therefore, $\angle GEH = \angle CEH$, or HE is the bisector of $\angle GEC$.

Therefore $CH : HG = CE : GE$.

But CE is greater than GE.

Therefore CH is greater than HG, or CH is greater than one-half of CG.

Draw GK and FL perpendicular to BC. $KC = \frac{1}{2} GC$.

Adding, $KC + CH$ is greater than CG. But GC is greater than GE.

Therefore $KC + CH$ is greater than GE.

Hence $AC + CK$ is greater than $AE + EG$.

Similarly, $AB+BL$ is greater than $AD+DF$. And, since $LK=FG$, the perimeter of $\triangle ABC$ is greater than the perimeter of $\triangle ADE$.

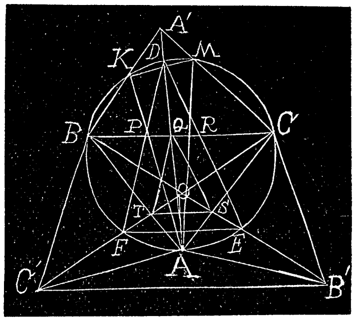
Excellent demonstrations were received from *P. H. PHILBRICK, G. B. M. ZERR, HENRY HEATON, C. A. LINDEMAN*, and *T. T. DAVIS*.

180. Proposed by R. TUCKER, M. A.

ABC is a triangle; A', B', C' are the images of A, B, C with respect to BC, CA, AB. The circum-circle ABC cuts $A'BC$ (say) in K (on $A'B$), M (on $A'C$) and AK, AM, AA' cut BC in P, R, Q, respectively. Prove that (1) the orthocenters of the associated triangles lie on circle ABC; (2) triangle AKM has its sides parallel to and equal twice the sides of the pedal triangle of ABC, and is also equal triangle formed by the above-named orthocenters; (3) $CP.a=b^2$, $BR.a=c^2$, $AP.a=AR.a=bc$, $BP.a=a^2-b^2$, $CR.a=a^2-c^2$, *i. e.*, $PR.a=2bc\cos A$, (4) hence BA touches circle ARC, which contains a Brocard-point of ABC; similarly for CA and circle APB; (5) $BR.CR'$, $AR''=abc=CP.BP'$, AP'' (where R', R'', P', P'' correspond to R, P, on CA, AB. respectively); K, K' are the Brocard constants ($k=a^2+b^2+c^2$) of ABB, $A'B'C'$; then $K'=K=\triangle^2/R^2$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

(1) Since the triangles $A'BC$, $B'AC$, $C'AB$ are equal to ABC, respectively, and A', B', C' are the images of A, B, C, the orthocenters of the triangles are the images of the orthocenter O, of the triangle ABC with respect to its sides.



But $BS.SE=AS.SC$, or $a\sin C.SE=a\cos C$. $c\cos A$. $\therefore$ $SE=c\cos A\cot C=SO$.

Similarly, $DQ=b\cos C\cot B=QO$, $TF=a\cos B\cot A=TO$.

$\therefore$ D, E, F are the orthocentres of the triangles.

(2) ArcKDC=arcCEA, both measured by $\angle B$.

ArcDC=arcCE, both measured by $\angle(\frac{1}{2}\pi-C)$.

$\therefore$ ArcKD=arcAE and DE is parallel to KA.

ArcMKB=arcBFA, both measured by $\angle C$.

ArcDB=arcBF, both measured by $\angle(\frac{1}{2}\pi-B)$.

$\therefore$ ArcDM=arcFA, and DF is parallel to AM.

$\therefore$ ArcKBA=arcDBF, arcDCE=arcMEA, arcKDM=arcFAE.

$\therefore$ $DF=KA=2QT$, $DE=MA=2QS$, $KM=FE=2TS$.

Also DE is parallel to AK is parallel to QS, DF is parallel to MA is parallel to QT, FE is parallel to TS. $\triangle DFE=\triangle AKM$ (three sides of one equal three sides of other).

(3) From triangle PAC, $\angle PAC=\angle B$, $\angle P=\angle A$.

$\therefore$ $CP\sin A=b\sin B$ or $CP.a=b^2$.

Similarly, from triangle BAR, $BR\sin A=c\sin C$, or $BR.a=c^2$, $\angle P=\angle R=\angle A$.

$\therefore AP = AR$ and $AP.a = AR.a$.

But $AP\sin A = b\sin C$ or $AP.a = bc = AR.a$.

$BP.a = (a - CP)a = a^2 - CP.a = a^2 - b^2$.

$CR.a = (a - BR)a = a^2 - BR.a = a^2 - c^2$.

$PR\sin A = AP\sin(\pi - 2A) = 2AP\sin A\cos A$.

$\therefore PR.a = 2AP.a\cos A = 2bc\cos A$.

(4) Since $BR.BC = AB^2$, AB touches the circle through ARC at A : therefore one of the Brocard points is on this circumference. Since $CP.CB = CA^2$, CA touches the circle through APB at A, which contains the other Brocard point.

(5) $BR = c^2/a$, $CR' = a^2/b$, $AR'' = b^2/c$, $CP = b^2/a$, $BP' = a^2/c$, $AP'' = c^2/b$.

$\therefore BR.CR'.AR'' = CP.BP'.AP'' = abc$; $(B'C')^2 = c^2 + b^3 - 2bc\cos 3A = a^2 + 8bc\cos A\sin^2 A$; $(A'C')^2 = b^2 + 8ac\cos B\sin^2 B$, $(A'B')^2 = c^2 + 8ab\cos C\sin^2 C$.

$$\begin{aligned}\therefore K' - K &= 8(bc\cos \mathrm{A}\sin^2 \mathrm{A} + ac\cos B\sin^2 B + ab\cos C\sin^2 C)\\ &= 32\triangle^2(\cos \mathrm{A}/bc + \cos B/ac + \cos C/ab)\\ &= (16\triangle^2/a^2b^2c^2)\Sigma(2a^2b^2 - a^4) = 256\triangle^4/a^2b^2c^2 = 16\triangle^2/R^2.\end{aligned}$$

181. Proposed by **WILLIAM HOOVER, A. M., Ph. D.**, Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Prove that the extremities of the latera recta of all ellipses having a given major axis $2a$ lie on the parabola $x^2 = -a(y-a)$.

Solution by **L. C. WALKER, A. M.**, Petaluma High School, Petaluma, Cal.; **J. R. HITT**, Coral Institute, San Marcos, Tex.; and the **PROPOSER**.

If (x_1, y_1) be an extremity of one of the latera recta, plainly, $y_1 = b^2/a$, or $b^2 = ay_1$....(1); also, $a^2 - b^2 = a^2e^2 = x_1^2$....(2), b and e having the usual meanings. Eliminating b from (1) and (2), $x_1^2 = -a(y_1 - a)$.

Also solved by *J. SCHEFFER*, and *G. B. M. ZERR*.

CALCULUS.

137. Proposed by **F. P. MATZ, Sc. D., Ph. D.**, Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Develop the equation of the curve assumed by the inextensible and revolving skipping rope.

No solution of this problem has been received.

138. Proposed by **M. E. GRABER, A. B.**, Tutor in Mathematics, Heidelberg University, Tiffin, O.

Find the curve the length of whose arc measured from a given point is a mean proportional between the ordinate and twice the abscissa.

Solution by **G. B. M. ZERR, A. M., Ph. D.**, Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa., and the **PROPOSER**.

From the problem, $s^2 = 2xy$ or $s = \sqrt{(2xy)}$.

$$ds = \sqrt{(dx^2 + dy^2)} = \frac{1}{\sqrt{2}}[\sqrt{(y/x)}dx + \sqrt{(x/y)}dy] \text{ or}$$

$$\sqrt{[1+(dy/dx)^2]}=\frac{1}{\sqrt{2}}[\sqrt{(y/x)}+\sqrt{(x/y)}(dy/dx)].$$

Let $y=mx$. $\therefore dy/dx=m+x(dm/dx)=m+px$.

$$\therefore \sqrt{[1+(m+px)^2]}=\frac{1}{\sqrt{2m}}(2m+px).\quad \frac{x^2(1-2m)p^2}{2m}+2x(1-m)p=(1-m)^2.$$

$$\therefore p=dm/dx=\frac{[1-m][1-\sqrt{(2m)}]\sqrt{(2m)}}{x[1-2m]}=\frac{\sqrt{[2m][1-m]}}{x[1+\sqrt{(2m)}]}.$$

$$\therefore dx/x=\frac{[1+\sqrt{(2m)}]\,dm}{[1-m]\sqrt{[2m]}}.$$

$$\therefore \log[Cx(1-m)]=\frac{1}{\sqrt{2}}\log\left[\frac{1+\sqrt{m}}{1-\sqrt{m}}\right].$$

$$\log[C(x-y)]=\frac{1}{\sqrt{2}}\log\left[\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}\right]=\frac{1}{\sqrt{2}}\log\left[\frac{x+y+2\sqrt{xy}}{x-y}\right].$$

For the given point $x=a$, $y=b$.

$$\therefore C=\frac{[a+b+2\sqrt{(ab)}]^{1/\sqrt{2}}}{[a-b][a-b]^{1/\sqrt{2}}}.$$

$$\therefore \frac{[x+y+2\sqrt{(xy)}]^{1/\sqrt{2}}}{[x-y][x-y]^{1/\sqrt{2}}}=\frac{[a+b+2\sqrt{(ab)}]^{1/\sqrt{2}}}{[a-b][a-b]^{1/\sqrt{2}}}.$$

$$\therefore \frac{a-b}{x-y}=\left[\frac{[x-y][a+b+2\sqrt{(ab)}]}{[a-b][x+y+2\sqrt{(xy)}]}\right]^{1/\sqrt{2}},\text{ or }[y-x]^{\sqrt{2}}=c\,\frac{\sqrt{y}+\sqrt{x}}{\sqrt{y}-\sqrt{x}}.$$

139. Proposed by **WM. FRED FLEMING**, Chicago. Ill.

A tin watering-pot is constructed by joining the frustums of two right cones, so that their intersection is a mathematical one, their axes meeting at an angle of 45°. The bases of the smaller frustum are 2 inches and 4 inches in diameter, its altitude 8 inches. The bases of the larger frustum are 10 inches and 12 inches in diameter, its altitude 15 inches. In joining the two frustums the edges of the two larger bases are brought into coincidence. Water is poured into the vessel until it begins to run out of the spout. How many gallons (231 cubic inches) are required ? How much water is in the spout and how much in the can? The vessel is tilted forward (in the plane of the axes of the two frustums) sufficiently to allow one-half of the water to run out. How much of the liquid is left in the spout and can, and what is the area of the surface of the water in spout and can ? Through what angle has the vessel been tilted ?

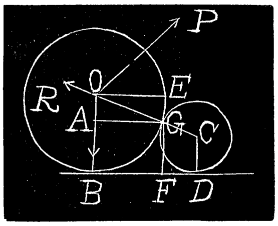
No solution of this problem has been received.

MECHANICS.

135. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy, Defiance College, Defiance, Ohio.

What force acting at an inclination ω with a horizontal line on the center of a wheel of given weight will roll the wheel over an immovable cylindric log whose diameter is $(1/m)$th that of the wheel?

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.



Let $CD=GC=a$, $OB=OE=ma$, P=force, R=reaction, W=weight of wheel, $\angle POE=\omega$, $\angle AOC=\theta$.

Resolving vertically, $W=R\cos\theta$.

Resolving horizontally, $P\cos\omega=R\sin\theta$.

$\therefore P\cos\omega/\sin\theta=W/\cos\theta$.

$\therefore P=W\tan\theta\sec\omega$.

$GF=a+a\cos\theta=a[1+\cos\theta]$.

$AO=ma\cos\theta=ma-a[1+\cos\theta]$.

$\therefore \cos\theta=[m-1]/[m+1]$. $\tan\theta=2\sqrt{m}/[m-1]$.

$\therefore P=\dfrac{2W\sqrt{(m)}\sec\omega}{m-1}$.

136. Proposed by F. T. WRIGHT, Ph. B., Schenectady, N. Y.

In an air brake test a train moving at 22 miles an hour on a down grade of one per cent. was stopped in 91 feet. There was 94 per cent. of the train braked. Taking the fractional resistance as 8 pounds per ton, find the net brake resistance per ton.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let the train weight T tons of 2240 lbs. The work due to gravity is $T(91\times2240)/100$. 22 miles per hour=$32\frac{4}{15}$ feet per second.

Let x=net brake resistance, g=32.16. Then

$$\frac{2240(32\frac{4}{15})^2T}{64.32}=8\times91T+\frac{94\times91Tx}{100}-\frac{91\times2240\,T}{100}.$$

$\therefore 36258.53=728+85.54x-2038.40$. $85.54x=37568.93$.

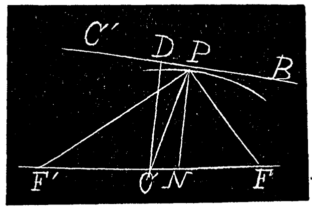
$x=439.2$ pounds per ton.

137. Proposed by G. B. M. ZERR, A. M., Ph. D., Profeseor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A uniform inextensible string rests against the inner side of a smooth elliptic wire semi-axes a and b, and is repelled from the foci and the center by the following forces: μ/rd and $\nu/r'd$ emanating from the foci, and $\pi c/d$ from the center, the distances of any point on the string from the foci being r and r', respectively, its distance from the center being c, and the semi-conjugate diameter corresponding to the point being d. Find the pressure on the wire at any point.

Solution by the PROPOSER.

Let P be any point, $C'B$ the tangent, PN the normal at P. $PF=r$, $PF'=r'$, $CP=c$, $\angle CPC'=\varphi$, $\angle FPB=\angle F'PC'=\theta$, $CD=p$, T=tension, n=normal force, m=tangential force, ρ=radius of curvature, R=pressure on wire at P. The equations of equilibrium are



$$dT/ds+m=0,\ T/\rho-R=n.$$

But $m=(\mu/rd)\cos\theta+(\nu/r'd)\cos\theta+(\pi c/d)\cos\varphi$.

$$n=\left(\frac{\mu}{r}+\frac{\nu}{r'}\right)\frac{\sin\theta}{d}+\frac{\pi c}{d}\sin\varphi.$$

But $\sin\theta=b/d$, $\sin\varphi=ab/dc$, $\cos\theta ds=dr=dr'$, $\cos\varphi ds=/dc$.

$$\therefore\ dT+\frac{\mu}{rd}dr+\frac{\nu}{r'd}dr'+\frac{\pi c}{d}dc=0\ldots.(1),$$

$$T/\rho-R=\left(\frac{\mu}{r}+\frac{\nu}{r'}\right)\frac{b}{d^2}+\frac{\pi ab}{d^2}\ldots.(2)$$

are the equations of equilibrium.

Now $d=\surd(rr')=\surd[r(2a-r')]=\surd[r'(2a-r)]=\surd[a^2+b^2-c^2]$.

$$\therefore\ dT+\frac{\mu dr}{r\surd[r(2a-r)]}+\frac{\nu dr'}{r'\surd[r'(2a-r')]}+\frac{\pi cdc}{\surd[a^2+b^2-c^2]}=0.$$

$$\text{Integrating, } T+\frac{\mu\surd[2a-r]}{a\surd r}-\frac{\nu\surd[2a-r']}{a\surd r'}-\pi\surd[a^2+b^2-c^2]=C,$$

$$\text{or } T-\frac{\mu}{a}\sqrt{\left(\frac{r'}{r}\right)}-\frac{\nu}{a}\sqrt{\left(\frac{r}{r'}\right)}-\pi d=C.$$

Let T_0 be the tension of the string at the extremity of the minor axis. Then $r=r'=a$, $d=a$.

$$\therefore\ C=\frac{aT_0-\mu-\nu-\pi a^2}{a}.$$

$$\therefore\ T=\frac{\mu}{a}\sqrt{\left(\frac{r'}{r}\right)}+\frac{\nu}{a}\sqrt{\left(\frac{r}{r'}\right)}+\pi d+\frac{aT_0-\mu-\nu-\pi a^2}{a}.$$

But $\rho=a^2b^2/p^3$ and $p=ab/d$.

$\therefore\ 1/\rho=p^3/a^2b^2=ab/d^3=ab/\surd[r^3r'^3]$.

$$\therefore\ T/\rho=\frac{b}{r^2r'^2}(\mu r'+\nu r+\pi arr')+\frac{aT_0-\mu-\nu-\pi a^2}{a\rho}.$$

$$\therefore\ \frac{b}{r^2r'^2}(\mu r'+\nu r+\pi arr')+\frac{aT_0-\mu-\nu-\pi a^2}{a\rho}-R=\frac{b}{r^2r'^2}(\mu r'+\nu r+\pi arr').$$

$$\therefore\ R=\frac{aT_0-\mu-\nu-\pi a^2}{a\rho}.$$

AVERAGE AND PROBABILITY.

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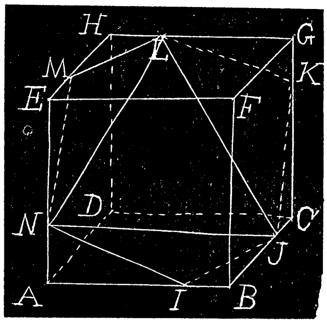
113. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy, Defiance College, Defiance. Ohio.

A given cube is cut by a plane in such a manner that the lines of section form a regular hexagon. What is the mean area of this hexagon?

Solution by G. B. M. ZERR. A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.



Let $ABCD-H$ be the cube, side a.

$HM=EN=AI=BJ=CK=GL=x$.

$LH=EM=AN=BI=CJ=GK=a-x$.

Then $IJKLMN$ is a hexagon.

$IJ=JK=KL=LM=MN=NI=\sqrt{(a^2+2x^2-2ax)}$.

$IK=JL=KM=LN=MI=NJ=\sqrt{(2a^2+2x^2-2ax)}$.

$\therefore$ The hexagon is a regular hexagon.

Area of this hexagon $=\frac{3}{2}(a^2+2x^2-2ax)\sqrt{3}=u$.

Average area $=\triangle=\int_{\frac{1}{2}a}^{a} u\,dx \Big/ \int_{\frac{1}{2}a}^{a} dx$.

$\therefore\ \triangle=\dfrac{3\sqrt{3}}{a}\int_{\frac{1}{2}a}^{a}(a^2+2x^2-2ax)dx=a^2\sqrt{3}$.

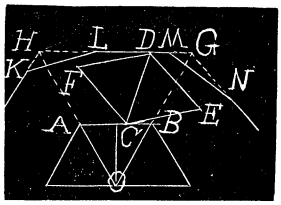
114. Proposed by L. C. WALKER, A. M., Professor of Mathematics, Petaluma High School, Petaluma, Cal.

If a regular polygon of n sides be placed at random on another equal polygon, show that the chance that the center of the first will fall on the second polygon is $\dfrac{\pi}{2[\pi+n\tan(\pi/n)]}$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia. Pa.

Let AOB, DCE be a triangle of each polygon. Let DCE move parallel to itself, then the center D will trace out the polygon of $2n$ sides $KLMGN$, etc.

Let $BO=CD=r$, $\angle BCE=\theta=$the angle BO makes with DC. Then the perpendicular distance from AB to $HG=r\cos[(\pi/n)-\theta]$, $HA=r\sec(\pi/n)\cos[(\pi/n)-\theta]=r[\cos\theta+\sin\theta\tan(\pi/n)]$, $OH=r[1+\cos\theta+\sin\theta\tan(\pi/n)]$.



$MG=\sqrt{(CD^2+BG^2-2CD.BG\cos\theta)}=r\sin\theta\sec(\pi/n)$.

$HG=2r[1+\cos\theta+\sin\theta\tan(\pi/n)]\sin(\pi/n)$.

$HL=HG-LG=r[2\cos\theta\sin(\pi/n)+2\sin\theta\sin(\pi/n)\tan(\pi/n)-\sin\theta\sec(\pi/n)]$.

Area $HOG=\frac{1}{2}r^2[1+\cos\theta+\sin\theta\tan(\pi/n)]^2\sin(2\pi/n)$.

Area $MGN=\frac{1}{2}r^2[2\sin\theta\cos\theta\tan(\pi/n)+\sin^2\theta\tan^2(\pi/n)-\sin^2\theta]\sin^2(\pi/n)$.

The total number of positions is $n(\triangle OGH-\triangle MGN)=nr^2\sin(2\pi/n)[1+\cos\theta+\sin\theta\tan(\pi/n)]=A$. The number of favorable positions is $n\triangle AOB=B$.

$\therefore B=\frac{1}{2}nr^2\sin(2\pi/n)$.

$$\therefore\ p=\int_0^{\pi/n}\beta d\theta \Big/ \int_0^{\pi/n} A d\theta=\tfrac{1}{2}\int_0^{\pi/n} d\theta \Big/ \int_0^{\pi/n}[1+\cos\theta+\sin\theta\tan(\pi/n)]d\theta=\frac{\pi}{2[\pi+n\tan(\pi/n)]}$$

Cor. When $n=\infty$, $p=\frac{1}{4}$; when $n=3$, $p=\pi/[2(\pi+3\sqrt{3})]$; when $\pi=4$, $p=\pi/[2(\pi+4)]$; when $n=6$, $p=\pi/[2(\pi+2\sqrt{3})]$.

BOOKS AND PERIODICALS.

Science of Mechanics. A critical and historical account of its development. By Dr. Ernst Mach, Professor of the History and Theory of Inductive Science in the University of Vienna. Translated from the German by Thomas J. McCormack. Second revised and enlarged edition, with 259 cuts and illustrations. 8vo. Cloth, xx+605 pages. Price, $2.00 net. Chicago: The Open Court Publishing Co.

This work differs from others on the same subject in treating of the principles of mechanics principally under the aspect of their development. To the student reading the subject for the first time, this method will prove easy and interesting. He will be able to master many principles at first reading that would require much study to master in other works. As a history of mechanics, it is excellent, and as a text-book on Mechanics it is admirable, its treatment being lucid, clear and forcible. It should be in the hands of every teacher of Physics and Mechanics.

Elementary Principles in Statical Mechanics. Developed with especial reference to the Rational Foundation of Thermodynamics. By J. Willard Gibbs, Ph. D., LL. D., Professor of Mathematical Physics in Yale University. 8vo. Cloth. xii+207 pages. Price, $4.00 net. New York: Charles Scribner's Sons.

The matter of this volume, the author says, consists in large measure of resulst which have been obtained by the investigations of Clausius, Maxwell, and Boltzmann, although the point of view and the arrangement is different. The first chapter considers the principle of conservation of extension-in-phase and derives what may be called the fundamental equation of statical mechanics. In the general case, the fundamental equation admits of integration, which gives a principle variously expressed, according to the point of view from which it is regarded, as the conservation of density-in-phase, or of extension-in-phase, or of probability-in-phase. This principle of the conservation of probability of phase is applied to the theory of errors in the calculated phase of a system, when the determination of the arbitrary constants of the integral equations are subject to error. In the third chapter, the principle of conservation of extension-in-phase is applied to the integration of the differential equation of motion. In the fourth and following chapters the author returns to the consideration of statical equilibrium and confines his attention to the conservative systems. In the fourteenth chapter thermodynamic is discussed, and in chapter fifteen is considered the modification of the preceding results which is necessary when systems composed of a number of entirely similar particles are considered.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single number, 25 cents. The Review of Reviews Co., 13 Astor Place, New York.

Among the contributed articles in the *Review of Reviews* for June are the following: "Bowdoin College: a Century of Service," by William I. Cole; "The Queen Regent and the Young King of Spain," by Helene Vacaresco; and "Two American Novelists" (Bret Harte and Frank Stockton).

The Literary Digest. A Weekly Compendium of the Contemporaneous Thought of the World. Price, $3.00 per year in advance. Single number, 10 cents. Funk & Wagnalls Co., Publishers, 30 Lafayette Place, New York.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited and Published by John Brisben Walker. Price, $1.00 per year in advance. Single numbers, 10 cents. Irvington-on-the-Hudson.

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. IX. AUGUST-SEPTEMBER, 1902. NOS. 8-9.

BIOGRAPHY.

CRISTOFORO ALASIA.

By DR. GEORGE BRUCE HALSTED, Austin, Texas.

When a new star comes out in the skies, thither turns the observing eye. "Le Matematiche" is a new luminary among scientific periodicals, though sailing safely now far into its second year.

Its director, its creator, Professor Alasia, has won the confidence and is attracting the attention of the mathematical world.

A sketch of his career cannot but be opportune, however brief and inadequate. That Professor Alasia is still very young to have won so prominent a position will be seen when we say, he was born in 1869, in Sassari, Sardinia.

His university course was carried on by turns in three different cities, Turin, Cagliari, and Rome. At Turin he was so fortunate as to have for masters those extraordinarily influential men, D'Ovidio and Peano.

At Rome he completed two courses at the School of Application for engineers under the direction of Senator Cremona. The course there in rational mechanics under Professor Cerruti is justly famous.

The sudden death of young Alasia's father recalled him to Sardinia. He won his first teaching position by competition. On this occasion he published his first book, on the theory of equations, (Naples, 1893).

CRISTOFORO ALASIA.

His love of science did not however prevent him from also occupying himself with the fine arts; he has won two prizes at expositions of dilettanti in painting. Professor Alasia is a gifted linguist. He has an elegant Italian style, writes the purest French, and at present is engaged with Professor Dionisio Gambioli in translating into Italian Cajori's History of Physics. They will enlarge the work by two additional chapters and copious notes. It is expected to appear at the end of this year.

In his Essay on the nomenclature (bibliographic) of the New Geometry of the triangle, our author proposes to give the most complete possible list of the terms which have entered the domain of geometry in these latter years, to give their veritable signification, to investigate what geometer has first used them, upon what occasion, etc.

In addition, when it is a question of a point or a straight, he has given its representation according to the method of Grassmann; and for circles and conics he has given the equation in barycentric or normal coördinates.

The monthly journal of pure and applied mathematics founded by Professor Alasia, "Le Matematiche," has had an extraordinary success.

The last thing ever written by the great Hermite was for it. Professor Alasia has been able to win the support and friendship of many of the most illustrious of living mathematicians, for example, Poincaré.

His is a charming figure in the new renaissance of creative productivity in Italy. His fine judgment and powers of assimilation are illustrated in his Poligeometrognomia generale e la Geometria Non-Euclidea del Chrystal, a translation of Chrystal's Non-Euclidean Geometry, preceded by a general resumé, historic and bibliographic, in exposition of the foundations of geometry, remarkable in erudition and breadth of insight.

His splendid fertility is amply shown in the subjoined list of his other writings:

Elementi della Teoria delle equagioni, ecc.—Napoli, 1893, B. Pellermo, ed.

Sulla deviazione dei gravi,—lettera alla Societa Astronomica d'Francia.

Su di alcune proprieta dei numeri e delle congruenze, ecc. Civitanova—Marche, 1898. C'est le vol. VI, sect. Science, de la Collection d'Audes italiane qui a été publiée par l'éditeur D. Natalucci.

Sulle involuzioni di ordini superiori,—Civitanova, 1898, Natalucci, ed.

Su di alcune proprieta delle linee geodetiche,—Sassari, 1898.

Esercize ed applicazioni di Calcolo infinitesimale e integrale—Citta di Castello, 1898, S. Lapi, ed.

Calcolo Grafico ed applicazioni alla Statica, ibid. 1899.

La Recente Geometria del triangolo, ibid. 1900.

566 Relazioni metriche e trigonometriche fra gli elementi d'un triangolo piano, —ibid. 1900.

Geometria e Trigonometria della Sfera, Milano, 1900, U. Hoepli, ed.

Esercizi ed applicazioni di Trigonometria piana, ibid. 1901.

Su di alcuni teoremi di Le Paige e Deruits, Madrid, 1899.

Producto de una serie cualqueira por la exponencial e^{-x}. (Estrait du Progreso Matemático de Zaragoza, ser. 2'. An. II, n°. 10, 1900).
Una trasformacion del Prof. Allardice (extr. ibid. n°. 11) (1900).
Alcune combinazioni di formule (extr. ibid. n°. 17) (1901).
Su di alcune proprieta delle Superfici a generatrice circolare (Supplem. al Vol. 1 delle Matematiche).
Una costruzione geometrica dell'equazione cubica (nel Vol. 1, n°. 5 del "Le Matematiche").
Une méthode élémentaire de recherche del maxima et minima. (Extr. de la Gazeta Matematica. An. VII, n°. 9, Mai 1902, Buçuresti,—(Rumania).
Saggio di nomenclatura della Recente Geometria del triangolo (dans le journal "Il Pitagora," vol. VIII, n'. 3a9, 1902).
Alcune asservazioni sui pendoli e sui cronometri (Extr. de la Rassegua Tecnica Italiana, An. II, n'. 4e5, Messina, 1902).
Complementi di Geometria elementare,—Milano, 1902, U. Hoepli, ed.
Elementi di Trigonometria Piana e Sferica, traduction, avec notej et adjointes du Traité de M. l'Abbé H. Gelin, de Huy.
Trattáto d Aritmetica, en collaboration de M. Gelīn, à Huy.
Alcune formule della Teoria delle Superficie (Extr. de la Revista simestral de Matemática, An. II, n°. 6, Zaragoza, 1902).

NINTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.*

By DR. L. E. DICKSON, of the University of Chicago.

The appropriateness of selecting a place as far west as Evanston, Illinois, for one of the Summer meetings of the whole Society was shown by the large and representative attendance as well as by the enthusiasm evinced at the four official sessions and at the several social concourses. Bearing on the geography of the subject is the fact that Chicago contributes the President of the Society, Professor E. H. Moore. There were present members from Columbia, Cornell, Johns Hopkins and other eastern institutions; from Kansas, the Dakotas, and other western States; while the middle west was very fully represented. The number of papers presented exceeded thirty, being equal to the number presented last Summer at Ithaca.

The program opened Tuesday morning, September second, with a paper by Dr. F. R. Moulton of the University of Chicago, entitled "A method of constructing general expressions for the elements of the planetary orbits which are valid for a finite time," in which objection was made to the so-called proofs by

*This report was written by request of the Editor.

the French astronomers of the past century of the stability of the solar system for infinite time.

Under certain restrictions, Dr. Moulton established rigorously the existence of periodic solutions. The important work of Dr. Moulton in introducing modern mathematical rigor in astronomical investigations cannot be too highly commended.

Professor Hathaway of Rose Polytechnic Institute, read a paper on "The quaternion treatment of the problem of three bodies," obtaining very condensed expressions for known results. In the discussion, several members expressed doubt as to the power of quaternion analysis for the discovery of new results.

Professor Dickson of the University of Chicago presented "Definitions of a linear associative algebra by independent postulates," as well as "Two definitions of a field by independent postulates;" giving also an account of the related paper by Dr. E. V. Huntington of Harvard University. Several members participated in a discussion of these three papers and the more genersl topic of axioms and postulates, as also illustrated by Hilbert's work in Geometry and the group definitions by Huntington and Moore. One point emphasized was the advantage of simple natural postulates, even if the *number* of independent postulates was larger than for a more artificial system. The three papers will appear in the *Transactions* of the Society.

The first paper on Tuesday afternoon was by Professor Oskar Bolza of the University of Chicago, who gave "Some instructive examples in the calculus of variations." To indicate the pedagogical value of the paper, it may be remarked that a volume by Professor Bolza on the Calculus of Variations is soon to appear in the Decennial Publications of the University of Chicago.

Professor J. B. Shaw of Kenyon College, read "On linear associative algebras," the paper being a continuation of the one read recently before the Chicago Section.

Dr. W. B. Fite of Cornell University, read "Concerning the commutator subgroups of groups whose orders are powers of primes."

Mr. J. W. Young of Cornell, presented a very clear preliminary report "On a certain group of linear substitutions and the functions belonging to it." The paper has close contact with Klein's Elliptic Modular Functions.

A paper on the topic "In technical theory of numbers," by Professor Westlund of Purdue University, was presented in abstract by Professor Moore.

Professor L. E. Dickson of Chicago, made a brief "Announcement of new simple groups," related to his paper in the *Transactions*, July, 1901.

On Wednesday morning, Professor Maschke gave a report on the paper "Ueber die Reducibilität der Gruppen linearer homogener Substitutionen" by Professor Alfred Loewy of the University of Freiburg. The paper will appear shortly in the *Transactions.*

Professor H. S. White of Northwestern University presented "A special twisted cubic with rectilinear directrix," generalizing certain interesting properties of conics.

Professor F. Morley of Johns Hopkins University applied the geometry of the complex variable to obtain "Orthocentric properties of the plane n-line." He announced theorems relating to a general system of straight lines.

Four other papers on the programme and read by title were of geometrical character; they were by Dr. Snyder of Cornell, Professor Field of Carthage College, Dr. Keyser of Columbia University, Professor Emch of Colorado University.

On Wednesday afternoon were presented several papers not on the printed programme, also (by title) papers by Dr. Eisenhart of Princeton, Professor Roe of Syracuse University, Dr. Epsteen of Philadelphia, Mr. Ford of the University of Michigan, and finally the following three papers:

"The bilinear point-line connex in space; an extension of Clebsch's connex," by Dr. E. Kasner of Columbia University; "Multiple points of Lissajous's curves in two and three dimensions," by Mr. E. A. Hook, Milwaukee; "Null systems in space of five dimensions," by Professor John Eiesland, Theil College.

A large number of the members lunched together each day at Evanston's one hotel, and dined together in one of the College buildings Tuesday evening, afterwards visiting the University observatory. On Thursday, the day after the Society adjourned, a number of members visisted the Chicago Steel Works and the University of Chicago, lunching together at the Quadrangle Club.

THE EXPRESSION OF THE nth POWER OF A NUMBER IN TERMS OF THE nth POWERS OF OTHER NUMBERS, n BEING ANY INTEGER; AND THE DEDUCTION OF SOME INTERESTING PROPERTIES OF PRIME NUMBERS.*

By J. W. NICHOLSON, A. M., LL. D., Professor of Mathematics in the Louisiana State University.

Any polynomial of m terms is equal to, and may be expressed under the form of:

+[The sum of its terms taken in sets of $m-1$]
−[" " " " " $m-2$]
+[" " " " " $m-3$]
−[" " " " " $m-4$]
. . . .

*This paper was read to the American Mathematical Society several years ago and in presenting it the author asked whether or not anything of the kind had ever before been discovered. The inquiry was referred to Dr. Artemas Martin who reported that the main formula, designated in this paper by (4), had been discovered and presented to the London Mathematical Society, but that the paper contained none of the transformations or deductions embodied in my discovery. With the hope that others will be interested in the matter and reach still more beautiful and important deductions, the paper as presented by me to the Society is now submitted to the public.

EXAMPLES.

1. $a_1+a_2+a_3=[(a_1+a_2)+(a_1+a_3)+(a_2+a_3)]-[(a_1)+(a_2)+(a_3)]\ldots.(1).$

2. $a_1+a_2+a_3+a_4=$

$$\left.\begin{array}{l}+[(a_1+a_2+a_3)+(a_1+a_2+a_4)+(a_1+a_3+a_4)+(a_2+a_3+a_4)]\\-[(a_1+a_2)+(a_1+a_3)+(a_1+a_4)+(a_2+a_3)+(a_2+a_4)+(a_3+a_4)]\\+[(a_1)+(a_2)+(a_3)+(a_4)]\end{array}\right\}\ldots.(2).$$

3. In general, making $s=a_1+a_2+\ldots.a_m$, we have

$$\left.\begin{array}{l}s=[(s-a_1)+(s-a_2)+\quad\cdot\quad\cdot\quad\cdot\quad(s-a_m)]\\\quad-[s-a_1-a_2)+(s-a_1-a_3)+\quad\cdot\quad\cdot\quad(s-a_{m-1}-a_m]\\\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\-(-1)^m[(a_1+a_2)+(a_1+a_3)+\quad\cdot\quad\cdot\quad\cdot\quad(a_{m-1}+a_m)]\\\quad(-1)^m[(a_1)+(a_2)+\ldots.(a_m)]\end{array}\right\}\ldots.(3).$$

That (3) is true may be shown thus:

Any term of s, as a_1, occurs in the first [ ] $m-1$ times; in the second [ ], $\frac{(m-1)(m-2)}{2}$ times; in the third, $\frac{(m-1)(m-2)(m-3)}{2.3}$ times; and so on to the last in which it occurs 1 time. Hence the sum of all the a_1's in the second member is

$$[(m-1)-\frac{(m-1)(m-2)}{2}+\ldots.(-1)^m]a_1=a_1.\qquad\text{Q. E. D.}$$

In all that follows, n is supposed to be a positive integer and less than m, unless otherwise stated.

An interesting and important property of polynomials arranged under the form of (3) is set forth in the following

PROPOSITION: *The second member of (3) may be raised to the nth power by raising each of the parts within the ( ) to that power, ignoring the signs before the* [ ]. Thus:

$$\left.\begin{array}{l}s^n=[(s-a_1)^n+(s-a_2)^n+\quad\cdot\quad\cdot\quad\cdot\quad(s-a_m)^n]\\\quad-[(s-a_1-a_2)^n+(s-a_1-a_3)^n+\quad\cdot\quad(s-a_{m-1}-a_m)^n]\\\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\-(-1)^m[(a_1+a_2)^n+(a_1+a_3)^n+\quad\cdot\quad\cdot\quad(a_{m-1}+a_m)^n]\\\quad(-1)^m[(a_1)^m+(a_2)^n+\ldots.(a_m)^n]\end{array}\right\}\ldots.(4).$$

PROOF. Let us suppose s^n, and each of the terms of the second member, to be expanded. Let $G=m'a_1{}^pa_2{}^q\ldots.a_k{}^z$ be any particular term of s^n, where $p+q+\ldots.z=n$, and k any integer from 1 to m. Now G occurs $m-k$ times in the

first [ ], $\frac{(m-k)(m-k-1)}{2}$ times in the second [ ], $\frac{(m-k)(m-k-1(m-k-2)}{2.3}$ times in the third [ ], and so on. Hence the sum of all the G's in the second member is

$$[m-k-\frac{(m-k)(m-k-1)}{2}+\ldots.(-1)^m]G=G, \qquad \text{Q. E. D.}$$

EXAMPLES.

1. Making $a_1=2$, $a_2=3$, $a_m=a_3=6$, whence $s=11$, and we have

$$11^n=9^n+8^n+5^n-6^n-3^n-2^n\ldots.(5),$$

where $n=2$ or 1.

2. Making $a_1=2$, $a_2=3$, $a_3=6$, $a_m=a_4=12$, we have

$$23^n=21^n+20^n+17^n+11^n-18^n-15^n-14^n-9^n-8^n-5^n+12^n+6^n+3^n+2^n\ldots.(6),$$

where $n=3$, 2 or 1.

3. Making $a_1=1$, $a_2=3$, $a_3=7$, $a_4=15$, $a_m=a_5=31$, we have

$$57^n=56^n+54^n-53^n+50^n-49^n-47^n+46^n+42^n-41^n-39^n+38^n-35^n+34^n+32^n$$
$$-31^n+26^n-25^n-23^n+22^n-19^n+18^n+16^n-15^n-11^n+10^n+8^n-7^n+4^n$$
$$-3^n-1^n\ldots.(7).$$

Here $n=4$, 3, 2 or 1.

Some additional properties of an interesting character are presented in the following corollaries to the preceding proposition.

COR. I. If $s=2^m-1$, then s^m may be expressed in terms of the nth powers of all the integers from 2^m-2 to 1, inclusive.

EXAMPLES.

1. For $a_1=1$, $a_2=2$, $a_m=a_3=4$, $s=7=2^3-1$, we have

$$(2^3-1)^n+5^n-4^n+3^n-2^n-1^n\ldots.(8),$$

where $n=2$ or 1.

2. For $a_1=1$, $a_2=2$, $a_3=4$, $a_m=a_4=8$, $s=15=2^4-1$, we have

$$(2^4-1)^n=14^n+13^n-12^n+11^n-10^n-9^n+8^n+7^n-6^n-5^n+4^n$$
$$-3^n+2^n+1^n\ldots.(9),$$

where $n=3$, 2 or 1.

NOTE.—By (8) we observe that the sum of the last seven terms of $(9)=0$. Hence,

$$15^n = 14^n + 13^n - 12^n + 11^n - 10^n - 9^n + 8^n \dots (10),$$

where $n=2$ or 1.

3. In general, to find $(2^m-1)^n$ we make $a_1=1$, $a_2=2$, $a_3=2^2$, $a_4=2^3, \dots$ $a_m=2^{m-1}$, and obtain

$$(2^m-1)^n = (2^m-2)^n + (2^m-3)^n - (2^m-4)^n \dots - (-1)^m 3^n$$
$$+ (-1)^m 2^n + (-1)^m \dots (11),$$

where $n=m-1$, $m-2$, ...2, or 1.

COR. II. If $s=(m-1)a_2 + \left(\frac{(m-1)(m-2)}{2} + 1\right)a_1$, then s^n may be expressed in terms of the nth powers of $2^m-2-2(m-2)^2$ integers, or less.

This is done by making $a_3=a_2+a_1$, $a_4=a_3+a_1$, $a_5=a_4+a_1$, etc., in (4), and eliminating the equal terms having contrary signs.

EXAMPLES.

1. Make $m=3$, $a_3=a_2+a_1$ and (4) becomes

$$(2a_2+2a_1)^n = (2a_2+a_1)^n + (a_2+2a_1)^n - a_2^2 - (a_1)^2 \dots (12),$$

where $n=2$ or 1.

Thus, for $a_2=7$, $a_1=5$, and $n=2$, we have

$$24^2 = 19^2 + 17^2 - 7^2 - 5^2 \dots (13).$$

2. Make $m=4$, $a_4=a_3+a_1$, $a_3=a_2+a_1$, and we have

$$(3a_2+4a_1)^n = (3a_2+3a_1)^n + (2a_2+4a_1)^n - (a_2+3a_1)^n$$
$$-(2a_2+a_1)^n + a_2^n + a_1^n \dots (14),$$

where $n=3$, 2, or 1.

Thus, for $a_2=7$, $a_1=5$, $n=3$, we have

$$41^3 = 36^3 + 34^3 - 22^3 - 19^3 + 7^3 + 5^3 \dots (15).$$

3. Make $m=5$, $a_5=a_4+a_1$, $a_4=a_3+a_1$, $a_3=a_2+1$, and we have

$$(4a_2+7a_1)^n = (4a_2+6a_1)^n + (3a_2+7a_1)^n - (2a_2+6a_1)^n - (3a_2+3a_1)^n$$
$$-(2a_2+4a_1)^n + (2a_2+3a_1)^n + (a_2+4a_1)^n + (2a_2+a_1)^n - a_2^n - a_1^n \dots (16),$$

where $n=4$, 3, 2, or 1.

4. Making $m=6$, $a_6=a_5+a_1$, $a_5=a_4+a_1$, etc., we have

$$(5a_2+11a_1)^n = (5a_2+10a_1)^n + (4a_2+11a_1)^n - (4a_2+6a_1)^n - (3a_2+10a_1)^n$$
$$-(3a_2+8a_1)^n + (3a_2+5a_1)^n + (2a_2+8a_1)^n + (3a_2+3a_1)^n$$

$$+(2a_2+6a_1)^n-(a_2+5a_1)^n-(2a_2+3a_1)^n-(2a_2+a_1)^n$$
$$=a_2{}^n+a_1{}^n \ldots.(17),$$

where $n=5$, 4, 3, 2, or 1.

Thus, making $a_2=5$, $a_1=2$, we have

$$47^n=45^n+42^n-35^n-32^n-31^n+26^n+25^n+2^n+5^n-12^n$$
$$-15^n-16^n+21^n+22^n \ldots.(18),$$

where $n=5$, 4, 3, 2, or 1.

Making $a_2=13$, $a_1=-5$, transposing, we have

$$24^n=22^n+21^n-15^n-13^n-12^n+10^n+5^n+4^n+3^n-1^n \ldots.(19),$$

where $n=5$, 3, or 1.

Making $a_2=8$, $a_1=-3$, transposing, we have

$$15^n=14^n+13^n-10^n-9^n+7^n-6^n+3^n+2^n+1^n \ldots.(20),$$

where $n=5$, 3, or 1.

COR. III. The root of each term in (4) may be increased by any number, as c, provided that the second member be increased by $-(-1)^m c^n$. Thus:

$$\begin{aligned}(s+c)^n= &+[(s+c-a_1)^n+(s+c-a_2)^n+ \quad . \quad . \quad . \quad (s+c-a_m)^n]\\ &-[(s+c-a_1-a_2)^n+(s+c-a_1-a_3)^n+ \ldots.(s+c-a_{m-1}-a_m)^n]\\ &\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \ldots.(21).\\ &-(-1)^m[(c+a_1+a_2)^n+(c+a_1+a_3)^n+ \quad . \quad . \quad (c+a_{m-1}+a_m)^n]\\ &(-1)^m[(c+a_1)^n+(c-a_2)^n+ \quad . \quad . \quad . \quad . \quad . \quad (c+a_m)^n]\\ &-(-1)^m c^n\end{aligned}$$

EXAMPLES.

1. Adding 8 to the roots of each term in (8) and we get (10).
2. Adding -5 to the roots of each term in (13) and we get

$$19^2=14^2+12^2+5^2-2^2 \ldots.(22).$$

3. Adding -22 to the roots of each term in (18), we get

$$25^n=23^n+22^n-17^n-13^n-9^n+7^n+6^n+4^n+3^n-1^n \ldots.(23),$$

where $n=5$, 3, or 1.

4. Adding -7 to the roots of each term in (19), we get

$$17^n=15^n+14^n-8^n+7^n-5^n-4^n-2^n \ldots.(24),$$

where $n=5$, 3, or 1.

In (21), by making $a_1=a_2=\dots.a_m$, we have

$$(c+ma_1)^n=m[c+(m-1)a_1]^n-\frac{m(m-1)}{2}[c+(m-2)a_1]^n+\dots-(-1)^m c^n\ .(25).$$

By (25) we may express the nth power of $c+ma_1$ in terms of the nth powers of the terms of the arithmetical progression

$$c+(m-1)a_1,\quad c+(m-2)a_1,\quad c+(m-3)a_1,\ \text{etc.},$$

multiplied by the binomial coefficients, m, $\frac{m(m-1)}{2}$, etc., respectively. Thus:

Making $c=5$, $m=6$, $a_1=7$, we have

$$47^n=6.40^n-15.33^n+20.26^n-15.19^n+6.12^n-5^n\ \dots.(26),$$

where $n=5$, 4, 3, 2, or 1.

Making $c=41$, $m=6$, $a_1=1$, we have

$$47^n=6.46^n-15.45^n+20.44^n-15.43^n+6.42^n-41^n\ \dots.(27).$$

Making $c=-15$, $m=6$, $a_1=6$, we have

$$21^n=6.15^n-15.9^n+20.3^n-15.(-3)^n+6(-9)^n-(-15)^n\ \dots.(28).$$

Hence, $21^n=5.15^n-9.9^n+5.3^n$, where $n=4$ or 2; and $21^n=7.15^n-21.9^n+35.3^n$, where $n=5$, 3, or 1.

Making $c=0$ and $a_1=1$, we have

$$m^n=m(m-1)^m-\frac{m(m-1)}{2}(m-2)^n+\dots.+(-1)^m m.1^n\dots.(29).$$

Thus, for $m=6$, we have $6^n=6.5^n-15.4^n+20.3^n-15.2^n+6$, where $n=5$, 4, 3, 2, or 1.

Cor. IV. Multiplying (4) by 2^n, and then adding $c-s$ to the root of each term, we have

$$\begin{aligned}(c+s)^n=&[(c+s-2a_1)^n+(c+s-2a_2)^n+\ \dots\ (c+s-2a_m)^n]\\ &-[(c+s-2a_1-2a_2)^n\ \dots\ (c+s-2a_{m-1}-2a_m)^n]\\ &\dots\dots\dots\dots.(30).\\ &-(-1)^m[(c-s+2a_1+2a_2)^n\ \dots\ [c-s+2a_{m-1}+2a_m)^n]\\ &(-1)^m[(c-s+2a_1)^n+(c-s+2a_2)^n+\ \dots\ (c-s+2a_m)^n]\\ &-(-1)^m(c-s)^n\end{aligned}$$

EXAMPLES.

Making $a_1=1$, $a_2=2$, $a_3=3$, $a_m=a_4=5$, $c=7$, we have

$$18^n=16^n+14^n-10^n-2^n\text{, or }9^n=8^n+7^n-5^n-1^n \ldots.(31),$$

where $n=3$ or 1.

Making $a_1=1$, $a_2=2$, $a_3=3$, $a_4=5$, $a_5=8$, $a_6=a_m=13$, we have

$$\begin{aligned}(c+32)^n=&(c+30)^n+(c+28)^n-(c+24)^n-(c+18)^n+(c+16)^n-(c+10)^n\\&+(c+8)^n+(c+6)^n-(c+4)^n-(c-4)^n+(c-6)^n+(c-8)^n \quad ..(32),\\&-(c-10)^n+(c-16)^n-(c-18)^n-(c-24)^n+(c-28)^n+(c-30)^n\\&-(c-32)^n\end{aligned}$$

where $n=5$, 4, 3, 2, or 1.

Making c, in (32), $=7$, we have

$$39^n=37^n+35^n-31^n+15^n+13^n-9^n-21^n \ldots.(33),$$

where $n=5$, 3, or 1.

In (30), by making $c=a_1=a_2=\ldots.a_m=1$, we have

$$(m-1)^n=m(m-1)^n-\frac{m(m-1)}{2}(m-3)^n+\frac{m(m-1)(m-2)}{6}(m-5)^n-\ldots.$$

$$\ldots.-(-1)^m\frac{m(m-1)}{2}(5-m)^n+(-1)^m m(3-m)^n-(-1)^m(1-m)^n.(34).$$

Supposing m and n to be both even or both odd, (34) may be written

$$(m+1)^n=(m-1)^{n+1}-\frac{m}{2}(m-3)^{n+1}+\frac{m(m-1)}{6}(m-5)^{n+1}$$

$$-\frac{m(m-1)(m-2)}{24}(m-7)^{n+1}+\ldots(35).$$

In (35), when m is even, the root of the last term is 1, and when the former is odd the latter is 2. Thus:

Making $m=6$, we have

$$7^n=5^{n+1}-3.3^{n+1}+5.1^{n+1} \ldots.(36),$$

where $n=4$ or 2.

[To be Continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

159. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

The amount of tax assessed on the property of a city is T,=\$145,850; and the treasurer was allowed a fee of $m\%$,=$\frac{3}{4}\%$, for collection. If $n\%$,=10%, of the tax was uncollectable, what were the net proceeds of the tax?

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College. Philadelphia, Pa., and J. R. HITT, Coronal Institute, San Marcos, Tex.

Net proceeds $=T(100\%-m\%)(100\%-n\%)$

$=\$145,850\times.90\times.99\frac{1}{4}=\$130,280.51\frac{1}{4}$.

If he was allowed $m\%=\frac{3}{4}\%$ of the whole, then net proceeds

$=T(100\%-m\%-n\%)$

$=\$145,850\times.89\frac{1}{4}=\$130,171.12\frac{1}{2}$.

160. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

A farm is rented for \$$R$,=\$300, in cash and a certain number of bushels of wheat. When wheat is \$$n$,=\$4-5 per bushel the rent is $p\%$,=$12\frac{1}{2}\%$ lower than when wheat is \$$m$, =\$1 1-5 per bushel. Find the number of bushels of wheat.

Solution by D. B. NORTHRUP, Mandana, N. Y.

Let x=the number of bushels of wheat. Then, by the conditions of the problem, $R+nx$=the number of dollars in the rent, when wheat is \$$n$ per bushel, and $R+mx$=the number of dollars in the rent when wheat is \$$m$ per bushel. But, also by the conditions of the problem, $R+nx=(1-p)(R+mx)$. Solving this equation for x, we find $x=\dfrac{pR}{(1-p)m-n}$. Substituting the numbers for the letters, $x=150$, the number of bushels of wheat.

ALGEBRA.

153. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Eliminate x, y, z, from the equations

$$x^2+yz=a,$$
$$y^2+zx=b,$$
$$z^2+xy=c,$$
$$x+y+z=0.$$

Solution by MARCUS BAKER, Washington, D C.

Let $s=a+b+c$; then

$$s+a=y^2+2yz+z^2+x(2x+y+z)=(y+z)^2+x^2=2x^2\,;$$
similarly, $s+b= \qquad =2y^2$;
$$s+c= \qquad =2z^2.$$

Hence $\sqrt{(s+a)}+\sqrt{(s+b)}+\sqrt{(s+c)}=(x+y+z)\sqrt{2}=0$, an equation involving only a, b, c.

If we wish to express this relation without radicals we transpose, square, and reduce; whence $a-s=\sqrt{[(s+b)(s+c)]}$, whence

$$s^2+as=a^2-bc$$
$$s^2+bs=b^2-ca$$
$$s^2+cs=c^2-ab.$$

Whence $4s^2=a^2+b^2+c^2-(ab+bc+ca)$; and therefore

$$a^2+b^2+c^2+3(ab+bc+ca)=0.$$

This may also be written

$$\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}+3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=0.$$

Example. Let $x=+1$, $y=+2$, $z=-3$; then $a=-5$, $b=+1$, $c=+11$.

Excellent solutions of this problem were received from *F. L. SAWYER*, *LON C. WALKER*, *JOSIAH H. DRUMMOND*, *J. K. ELLWOOD*, *J. SCHEFFER*, and *G. B. M. ZERR*.

Mr. Baker sent in neat solutions of problems 151 and 152.

154. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Show that the equation, $x^4+qx^2+s=0\ldots.(1)$, can not have three *equal* roots.

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, Ohio.

By the usual theory, the conditions that $ax^4+4bx^3+6cx^2+4dx+e=0..(1)$ shall have three equal roots are

$$ae-4bd+3c^2=0\ldots.(2),$$
$$ad^2+eb^2+c^3-ace-2bcd=0\ldots.(3).$$

In the given equation, $x^4+qx^2+s=0\ldots.(4)$, $a=1$, $b=0$, $c=\frac{1}{6}q$, $d=0$, $e=s$, and (2) and (3) become

$$s+\frac{q^2}{12}=0\ldots.(5),\quad \frac{q^3}{36}-qs=0\ldots.(6).$$

Eliminating s from (5) and (6) gives $q=0$, which is inconsistent with the supposition.

Also solved similarly by *G. B. M. ZERR, F. L. SAWYER, JOSIAH H. DRUMMOMD, J. R. HITT,* and *HARRY S. VANDIVER.*

155. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

If the roots of the cubic $x^2+3px^2+3qx+r=0$ be in harmonical progression, $2q^3=r(3pq-r)$.

Solution by HOMER R. HIGLEY. M. S., State Normal School, East Stroudsburg, Pa.; J. R. HITT, Coronal Institute, San Marcos. Tex.; H. S. VANDIVER, Bala, Pa.; and the PROPOSER.

If α, β, γ be the roots of a cubic, we have

$$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\alpha\gamma+\beta\gamma)x-\alpha\beta\gamma=0....(1).$$

By the condition of the problem,

$$\beta=\frac{2\alpha\gamma}{\alpha+\gamma},\text{ or } \alpha\beta+\alpha\gamma+\beta\gamma=3\alpha\gamma....(2),$$

and (1) becomes

$$x^3-(\alpha+\beta+\gamma)x^2+3\alpha\gamma x-\alpha\beta\gamma=0....(3).$$

Comparing (3) and

$$x^3+3px^2+3qx+r=0....(4),$$

$$\alpha+\beta+\gamma=-3p....(5),\quad \alpha\gamma=q....(6),\quad \text{and } \alpha\beta\gamma=-r....(7).$$

From (6) and (7), $\beta=-r/q....(8)$.

From (5) and (8), $\alpha+\gamma=r/q-3p....(9)$.

Eliminating α, β, γ from (2) by use of (6), (8), (9), $2q^3=r(3pq-r)$.

Also solved by *J. SCHEFFER, LON C. WALKER, JOSIAH H. DRUMMOND, F. L. SAWYER,* and *G. B. M. ZERR.*

GEOMETRY.

183. Proposed by S. F. NORRIS. Professor of Mathematics and Astronomy in Baltimore City College, Baltimore, Md.

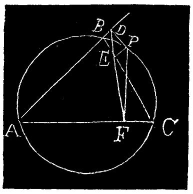
Two quadrilaterals having three sides of the one equal to the three corresponding sides of the other, each to each, and the two corresponding angles adjacent to the unknown sides equal, each to each, are equal figures. [*Olney's Geometry*, page 129.]

This is problem 169, an erroneous demonstration of which was published in the April number of the current volume of the MONTHLY. The fallacy in the demonstration is pointed out in Dr. Halsted's article, "Proving the False," on page 129. ED.

184. Proposed by ERWIN MARTIN, Principal of Schools, Mead, Neb.

If from any point in the circumference of a circle circumscribed about a triangle, perpendiculars are drawn to the sides, or the sides produced, of the inscribed triangle, the lines connecting the feet of the perpendiculars are collinear.

Demonstration by J. R. NITT, Coronal Institute, San Marcos, Tex.; G. I. HOPKINS, A.M., High School, Manchester, N. H.; and G. W. GREENWOOD, B. A., McKendree College, Lebanon, Ill.



Let ABC be a triangle; PD, PE, PF perpendiculars from any point P of circumcircle upon the sides.

To prove D, E, F collinear: Draw BP and CP, DE and DF. The quadrilaterals $BEPD$, $CFPD$ are cyclic.

$\therefore$ $\angle PDE=\angle PBE$, $\angle PDF=\angle PCF$. But $\angle PBE=\angle PCF$, each being measured by $\frac{1}{2}$arcAP.

$\therefore$ $\angle PDE=\angle PDF$. $\therefore$ D, E, F are collinear.

Also demonstrated by *BEULAH FRAZIER*, Soph., Rolla School of Mines, and *G. B. M. ZERR*.

162. Proposed by J. D. PALMER, Providence, Ky.

Given the distances from the vertices of a triangle, ABC, to the center of the incircle, to construct the triangle.

Solution by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

Already two solutions of this problem have appeared in the MONTHLY, an erroneous one in the January number, page 15, and a revised solution in the February number, pages 45-46, yet a word or two more may not be superfluous.

The problem is to *construct* a triangle knowing the distances from the incenter to the vertices.

The correct analysis of the problem, as Zerr shows in his revised solution results in a cubic equation, which means that the problem is insoluble, *i. e.* insoluble in the same sense that the trisection of an angle is impossible. The true answer is that the construction called for can not be made by elementry geometry.

The following is a trigonometrical analysis of the problem.

If A, B, C are the angles of a plane triangle, then

$$2\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C+\sin^2\tfrac{1}{2}A+\sin^2\tfrac{1}{2}B+\sin^2\tfrac{1}{2}C=1\ldots.(1).$$

This theorem readily results from substituting in the following well known relation

$$\cos A+\cos B+\cos C-1=4\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C$$

for $\cos A$, $1-2\sin^2\frac{1}{2}A$, etc., and reducing.

Now let α, β, γ be the distances from the incenter to the vertices and r the radius of the inscribed circle; whence

$$\sin\tfrac{1}{2}A=r/\alpha,\quad \sin\tfrac{1}{2}B=r/\beta,\quad \sin\tfrac{1}{2}C=r/\gamma.$$

Substituting in (1), we have

$$\frac{2r^3}{\alpha\beta\gamma}+\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}\right)r^2=1, \text{ or } r^3+\tfrac{1}{2}\left(\frac{\alpha\beta}{\delta}+\frac{\beta\gamma}{\alpha}+\frac{\gamma\alpha}{\beta}\right)r^2=\tfrac{1}{2}\alpha\beta\gamma.$$

Example. Let $\alpha=275$, $\beta=325$, $\gamma=429$; then $r^3+539\frac{1}{6}r^2-19170937\frac{1}{2}=0$; whence $(r-165)(r^2+704\frac{1}{6}r+116187\frac{1}{2})=0$.

$\therefore r=165$. Also, $r=-137.97+$ and $r=842.13+$.

CALCULUS.

140. Proposed by C. C. BEBOUT, Professor of Mathematics, Elgin High School, Elgin, Ill.

A pole two inches in diameter is set vertically in a level plat of ground. At a point ten feet from the ground a string is attached. A man holds the other end of the string and walks about the pole keeping the string stretched taut, and his hand at a constant distance of four feet from the ground, till the string is all wound upon the pole. If string is ten feet long, how far has his hand moved in the operation?

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let n be the number of times the string 10 feet feet in length will encircle the cylinder in a height of 6 feet. Circumference of cylinder$=\frac{1}{6}\pi$ feet, x=distance between successive portions of the string on the same generating line.

Then $\sqrt{[(\frac{1}{6}\pi)^2+x^2]}=10/n$ also $nx=6$.

$$\therefore n\sqrt{\left(\frac{\pi}{6}\right)^2+\left(\frac{6}{n}\right)^2}=10 \text{ or } \frac{\pi^2n^2}{36}=64. \quad \therefore n=\frac{48}{\pi},\ x=\tfrac{1}{8}\pi.$$

The distance moved by the hand is given on page 319, No. 9, Vol. I, of the MONTHLY, and is there worked out in full. It is $s=2\pi^2r^2m^2$ where r is the radius of the cylinder and m is the number of times the string is wound around the cylinder or the same as the value of n above. $r=\frac{1}{12}$ feet, $m=n=48/\pi$.

$\therefore s=32$ feet.

Also solved by *T. T. DAVIS*.

141. Proposed by B. F. FINKEL, A.M.. M.Sc.. Professor of Mathematics and Physics, Drury College, Springfield, Mo.

The curve $r^n=a^n\sin n\theta$ rolls along a straight line. Show that the intrinsic equation to the evolute of the locus of the pole is $s^n=a^m[1+1/n]^n\sin\psi$. [Edward's *Differential Calculas*, page 502.

Solution by WILLIAM HOOVER, A. M.. Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The distance from the point of tangency to the pole is normal to the path of the pole. Let this distance, which is a radius vector of the rolling curve, be r, and the perpendicular from the pole upon the fixed line be p.

Taking the fixed line as the axis of x, p is an ordinate of the pole, and r is the length of normal and is expressed by

$$y\sqrt{1+\frac{dy^2}{dx^2}}.$$

Now, all curves of the form $r^m=a^m\cos m\theta$ have their pedal equations in the form $r^{m+1}=a^m p....[a]$.

It is plain that the given curve may be so expressed. The equation $[a]$ then gives for the differential equation of the locus of the pole

$$\left(y\sqrt{1+\frac{dy^2}{dx^2}}\right)^{n+1}=a^n y, \text{ or, } \frac{dy}{dx}=\surd[a^{2n/(n+1)}-y^{2n/(n+1)}\div y^{n/(n+1)},$$

in which $dy/dx=\cot\varphi$ from the form of the given curve.

Then $y=a\sin^{(n+1)/n}\phi$, $\dfrac{dy}{d\phi}=\dfrac{n+1}{n}a\sin^{1/n}\phi\cos\phi....[b]$.

Also, $\dfrac{dx}{dy}=\tan\phi$, $\dfrac{ds}{dy}=\dfrac{1}{\cos\phi}$; then $\dfrac{ds}{d\phi}=\dfrac{n+1}{n}a\sin^{1/n}\phi....[c]$.

But this gives the radius of curvature in the locus of the pole, which again is the length of arc corresponding in the evolute, which vanishes with ϕ.

$\therefore$ the evolute is $s=\dfrac{n+1}{n}a\sin^{1/n}\phi$, as required.

Also solved by *G. B. M. ZERR.*

142. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Solve the differential equation,

$$[a-x]\frac{dz}{dx}+[b-y]\frac{dz}{dy}=c-z.$$

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; WILLIAM HOOVER, A.M., Ph. D., Ohio University, Athens, O.; and the PROPOSER.

Putting, as usual, $dz/dx=p$, $dz/dy=q$, and then comparing the given equation with $Pp+Qq=R$, we have $P=a-x$, $Q=b-y$, $R=c-z$.

Then, $\dfrac{dx}{P}=\dfrac{dz}{R}$ gives $\dfrac{dx}{x-a}=\dfrac{dz}{z-c}$, and

$$\frac{dy}{Q}=\frac{dz}{R} \text{ gives } \frac{dy}{y-b}=\frac{dz}{z-c}.$$

These equations give $\dfrac{x-a}{z-c}=\alpha$, $\dfrac{y-b}{z-c}=\beta$; so that the solution is

$$\frac{b-y}{c-z}=\phi\left(\frac{a-x}{c-z}\right).$$

Solved similarly by *LON C. WALKER.*

143. Proposed by L. C. WALKER, A. M., Petaluma High School, Petaluma, Cal.

Find the area of greatest ellipse that can be inscribed in a given semicircle.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; J. SCHEFLER, A. M., Hagerstown, Md.; and the PROPOSER.

Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ldots.(1),$$

and of the circle, $x^2+(y+b)^2=r^2\ldots.(2)$.

By eliminating x^2 from (1) and (2), then writing the condition for equal roots, we have

$$b^2=a^2-\frac{a^4}{r^2}\ldots.(3).$$

Required area$=\pi ab$, or say $u=a^2b^2=a^4-(a^6/r^2)$. Hence for a maximum, we get

$$4a^3=\frac{6a^5}{r^2}, \text{ or } a^2=\frac{2r^2}{3}. \text{ Then the area}=\frac{2\sqrt{3}}{9}\pi r^2.$$

Also solved by *JOSIAH H. DRUMMOND.*

MECHANICS.

138. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, Gloucestershire, England.

A smooth elliptical tube is held in the vertical plane with its major axis inclined to the vertical. A particle is projected from the lowest point. Find the pressure on the tube at any point and the condition that the pressure may vanish at the highest point.

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The reaction on the tube is given by

$$v^2/\rho=R+X\rho(d^2x/ds^2)+Y\rho(d^2y/ds^2)=R+X(dy/ds)+Y(dx/ds).$$

Let the x-axis be vertical and let β be the inclination of the major axis to the vertical; then

$$(a^2\sin^2\beta+b^2\cos^2\beta)x^2+(a^2\cos^2\beta+b^2\sin^2\beta)y^2+2(a^2-b^2)xy\sin\beta\cos\beta=a^2b^2$$

or $Ax^2+By^2+2Cxy=a^2b^2$, is the equation to the tube.

But $x=r\cos\theta$, $y=r\sin\theta$.

$$\therefore r=\frac{ab}{\sqrt{(A\cos^2\theta+B\sin^2\theta+2C\sin\theta\cos\theta)}}$$

$$=\frac{ab\sqrt{2}}{\sqrt{[A+B+(A-B)\cos 2\theta+2B\sin 2\theta]}}=\frac{ab\sqrt{2}}{D}.$$

$$dr=\frac{ab\sqrt{2}[(A-B)\sin 2\theta-2C\cos 2\theta]d\theta}{D^3}=\frac{ab\sqrt{2}Ed\theta}{D^3}.$$

$$ds=\frac{2ab\sqrt{[A^2+B^2+2C^2+(A^2-B^2)\cos 2\theta+2C(A+B)\sin 2\theta]}}{D^2}d\theta$$

$$=\frac{2abFd\theta}{D^3},\quad dy=dr\sin\theta+r\cos\theta d\theta.$$

Now $X=-g$, $Y=0$. $\therefore R=v^2/\rho+g(dy/ds)$.

$$\frac{dy}{ds}=\frac{E\sin\theta+D^2\cos\theta}{F\sqrt{2}}.\quad \therefore R=\frac{v^2}{\rho}+\frac{g(E\sin\theta+D^2\cos\theta)}{F\sqrt{2}}.$$

When $R=0$ and $\theta=\pi$, $v^2=\frac{Ag\rho}{\sqrt{(A^2+C^2)}}=\frac{g(Aa^2+Ab^2-a^2b^2)^{\frac{3}{2}}}{ab\sqrt{(A^3+AC^2)}}$.

If $\beta=0$, $v^2=b^2g/a$; if $a=b$, $v^2=ag$.

DIOPHANTINE ANALYSIS.

92. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find the sides of integral right triangles when the difference of the legs is given.

No solution of this problem has been received.

93. Proposed by the late SYLVESTER ROBBINS.

Solve and set forth twenty terms in some infinite series of rational parallelopipeds following the solid whose edges are 2, 3, 6, and diagonal 7.

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let x, $x+1$, x^2+x be the edges. Then $x^2+(x+1)^2+(x^2+x)^2=(x^2+x+1)^2$. $\therefore$ The diagonal$=x^2+x+1$.

No.	x	$x+1$	x^2+x	x^2+x+1
1	2	3	6	7
2	3	4	12	13
3	4	5	20	21
4	5	6	30	31
5	6	7	42	43
6	7	8	56	57
7	8	9	72	73

8	9	10	90	91
9	10	11	110	111
10	11	12	132	133
11	12	13	156	157
12	13	14	182	183
13	14	15	210	211
14	15	16	240	241
15	16	17	272	273
16	17	18	306	307
17	18	19	342	343
18	19	20	380	381
19	20	21	420	421
20	21	22	462	463
etc.	etc.	etc.	etc.	etc.

94. Proposed by L. C. WALKER, A. M., Petaluma High School. Petaluma, Cal.

Show that the area of a rational triangle cannot be a square number.

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let a^2+b^2, a^2+c^2, b^2+c^2 be the sides.

$\therefore$ Area$=abc\sqrt{(a^2+b^2+c^2)}$.

Let $a=m^2+mn$, $b=mn+n^2$, $c=mn$.

$\therefore$ Area$=mn(m^2+mn)(n^2+mn)(m^2+n^2+mn)$.

Let $n=1$. $\therefore$ Area$=m^2(m+1)^2(m^2+m+1)$.

For integral values of m, m^2+m+1 is not a square.

Let $9p$, $10p$, $17p$ be the sides. Then area$=36p^2$.

Let $3p$, $25p$, $26p$ be the sides. Then area$=36p^2$.

This gives two series of triangles whose areas are square numbers, thus proving that the proposition is not true.

95. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

There are two unequal square numbers the sum of whose sum, difference, product, and quotient, is a square. Find the two numbers.

This is problem 91, so numbered by mistake. Its solution is in the April number, page 113. ED.

AVERAGE AND PROBABILITY.

115. Proposed by L. C. WALKER, A. M., Petaluma High School, Petaluma, Cal.

Three points are at random within a given triangle. Find the chance that they will all lie on one side of some one line that can be drawn through the center of gravity of the triangle.

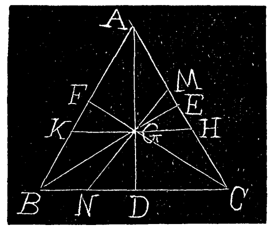
Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and J. R. HITT, Coronal Institute, San Marcos, Tex.

Project the given triangle into an equilateral triangle, side$=a$; let G be the center of gravity, CH or $CM=x$.

Then $BK=\dfrac{a(a-2x)}{2a-3x}$, $AK=\dfrac{a(a-x)}{2a-3x}$, $AH=a-x$.

Area $AKH=\dfrac{\triangle(a-x)^2}{a(2a-3x)}$, $BN=\dfrac{2ax-a^2}{3x-a}$,

$$CN=\frac{ax}{3x-a}.$$



Area $CMN=\dfrac{\triangle x^2}{a(3x-a)}$, area $ABNM=\triangle\left[1-\dfrac{x^2}{a(3x-a)}\right]$.

$$p=\left[\int_0^{\frac{1}{2}a}\frac{(AKH)^3dx}{\triangle^3}+\int_{\frac{1}{2}a}^{a}\frac{(ABNM)^3dx}{\triangle^3}\right]\Big/\int_0^{\frac{1}{2}a}dx$$

$$=\frac{2}{a}\int_{\frac{1}{2}a}^{a}\left[1-\frac{x^2}{a(3x-a)}\right]^3dx+\frac{2}{a^4}\int_0^{\frac{1}{2}a}\frac{(a-x)^6dx}{(2a-3x)^3}$$

$$=\frac{2}{a}\int_{\frac{1}{2}a}^{a}\left[1-\frac{x^2}{a(3x-a)}\right]^3dx+\frac{2}{a^4}\int_{\frac{1}{2}a}^{a}\frac{x^6dx}{(3x-a)^3}$$

$$=\frac{2}{a}\int_{\frac{1}{2}a}^{a}dx-\frac{6}{a^2}\int_{\frac{1}{2}a}^{a}\frac{x^2dx}{3x-a}+\frac{6}{a^3}\int_{\frac{1}{2}a}^{a}\frac{x^4dx}{(3x-a)^2}$$

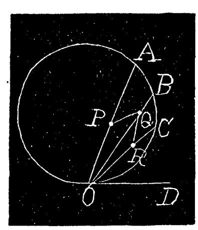
$$=\tfrac{1}{27}(\tfrac{23}{2}-\tfrac{20}{3}\log 2).$$

116. Proposed by the late ENOCH BEERY SEITZ.

The average area of the quadrilateral formed by joining four random points on the surface of a circle, radius a, is $4a^2/3\pi$.

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics. The Temple College, Philadelphia, Pa.

Let a=radius of given circle, A=its area, $\triangle$=required average, $\triangle'$=the average area when the four points are taken on both the circle A and a concentric annulus B, $\triangle_1$=the average area when three points are taken on A and one on B.



Then $(\triangle'-\triangle)A=4B(\triangle_1-\triangle)$.

But $\triangle:\triangle'=A:A+B$.

$\therefore\ \triangle'=\dfrac{(A+B)\triangle}{A}$. $\therefore\ \triangle=\frac{4}{5}\triangle_1$.

Let one point O be on the circumference of the circle, and the other points P, Q, R anywhere on its surface. Let $OP=x$, $OR=y$, OQ

$=z$, $\angle AOD=\theta$, $\angle COD=\phi$, $\angle BOD=\psi$. Then $OA=2a\sin\theta=x'$, $OC=2a\sin\phi=y'$, $OB=2a\sin\psi=z'$. Area $OPQR=\frac{1}{2}xz\sin(\theta-\psi)+\frac{1}{2}yz\sin(\psi-\phi)=u$.

$$\therefore\ \triangle_1=\frac{\int_0^\pi\int_0^\theta\int_\phi^\theta\int_0^{x'}\int_0^{y'}\int_0^{z'} uxyzd\theta d\phi d\psi dxdydz}{\int_0^\pi\int_0^\theta\int_\phi^\theta\int_0^{x'}\int_0^{y'}\int_0^{z'} xyzd\theta d\phi d\psi dxdydz}$$

$$\therefore\ \triangle=\tfrac{4}{5}\triangle_1=\frac{24}{5\pi^3a^6}\int_0^\pi\int_0^\theta\int_\phi^\theta\int_0^{x'}\int_0^{y'}\int_0^{z'} uxyzd\theta d\phi d\psi dxdydz$$

$$=\frac{32}{5\pi^3a^3}\int_0^\pi\int_0^\theta\int_\phi^\theta\int_0^{x'}\int_0^{y'}[x\sin(\theta-\psi)+y\sin(\psi-\phi)]xy\sin^3\psi d\theta d\phi d\psi dxdy$$

$$=\frac{64}{15\pi^3a}\int_0^\pi\int_0^\theta\int_\psi^\theta\int_0^{x'}[3x\sin(\theta-\psi)+4a\sin\phi\sin(\psi-\phi)]\sin^2\phi\sin^3\psi d\theta d\phi d\psi xdx$$

$$=\frac{512a^2}{15\pi^3}\int_0^\pi\int_0^\theta\int_\phi^\theta[\sin\theta\sin(\theta-\psi)+\sin\phi\sin(\psi-\phi)]\sin^2\theta\sin^2\phi\sin^3\psi d\theta d\phi d\psi$$

$$=\frac{64a^2}{15\pi^2}\int_0^\pi\int_0^\theta[3(\phi-\theta)(\sin\theta\cos\theta-\sin\phi\cos\phi)+3(\sin\theta\cos\theta-\sin\phi\cos\phi)^2$$

$$+2\sin(\theta-\phi)(\sin^3\theta\cos\phi-\cos\theta\sin^3\phi)]\sin^2\theta\sin^2\phi d\theta d\phi$$

$$=\frac{2a^2}{45\pi^3}\int_0^\pi(105\theta-72\theta^2\sin\theta\cos\theta+84\theta\sin^2\theta-120\theta\sin^4\theta-105\sin\theta\cos\theta-82\sin^3\theta\cos\theta$$

$$+32\sin^5\theta\cos\theta)\sin^2\theta d\theta=4a^2/3\pi.$$

MISCELLANEOUS.

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111. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Exhibit $\cos^3\theta\sin^3\theta\sin^2\phi\cos\phi$ as a series of harmonics.

Solution by the PROPESER.

$$f(\mu,\ \phi)=\sum_{m=0}^{m=\infty}[A_{0,m}P_m(\mu)+\sum_{n=1}^{n=m}(A_{n,m}\cos n\phi+B_{n,m}\sin n\phi)P_m^{\,n}(\mu)].$$

Now $f(\mu,\ \phi)=\mu^3\sqrt{[(1-\mu^2)^3]}\sin^2\phi\cos\phi=\frac{1}{4}\mu^3(1-\mu^2)^{\frac{3}{2}}(\cos\phi-\cos3\phi)$.

$$A_{0,m}=\frac{2m+1}{16\pi}\int_{-1}^1\mu^3(1-\mu^2)^{\frac{3}{2}}P_m(\mu)d\mu\int_0^{2\pi}(\cos\phi-\cos3\phi)=0.$$

$$B_{n,m}=\frac{2m+1}{8\pi}\cdot\frac{(m-n)!}{(m+n)!}\int_{-1}^{1}\mu^3(1-\mu^2)^{\frac{3}{2}}\,P_m{}^n(\mu)d\mu\int_0^{2\pi}(\cos\phi-\cos3\phi)$$

$$\times\sin n\phi d\phi=0.$$

$$A_{n,m}=\frac{2m+1}{8\pi}\cdot\frac{(m-n)!}{(m+n)!}\int_{-1}^{1}\mu^3(1-u^2)^{\frac{3}{2}}\,P_m{}^n(\mu)d\mu\int_0^{2\pi}\cos\phi\cos n\phi d\phi=0,$$

unless $n=1$.

If $n=1$, $\int_0^{2\pi}\cos\phi\cos n\phi d\phi=\int_0^{2\pi}\cos^2\phi d\phi=\pi$.

$$A_{1,m}=\frac{2m+1}{8}\cdot\frac{(m-1)!}{(m+1)!}\int_{-1}^{1}\mu^3(1-\mu^2)^2\,\frac{dP_m(\mu)}{d\mu}d\mu$$

$$=\frac{1}{2^m.m!}\cdot\frac{2m+1}{8}\cdot\frac{(m-1)!}{(m+1)!}\int_{-1}^{1}\mu^3(1-\mu^2)^2\frac{d^{m+1}(\mu^2-1)^m}{(d\mu)^{m+1}}d\mu.$$

$$\int_{-1}^{1}\mu^3(1-\mu^2)^2\,\frac{d^{m+1}(\mu^2-1)^m}{(d\mu)^{m+1}}d\mu=-5040\int_{-1}^{1}\frac{d^{m-6}(\mu^2-1)^m}{(d\mu)^{m-6}}d\mu$$

by repeated integration by parts, $=0$ if $m>6$,

$$=-5040\int_{-1}^{1}(\mu^2-1)^6d\mu=-\tfrac{491520}{143},\text{ if } m=6.$$

$$\therefore\ A_{1,\,6}=-\frac{1}{2^6.6!}\cdot\frac{13}{8}\cdot\frac{5!}{7!}\cdot\frac{491520}{142}=-\tfrac{2}{693}.$$

Similarly, $A_{1,\,5}=0$, $A_{1,\,4}=\frac{1}{770}$, $A_{1,\,3}=0$, $A_{1,\,2}=\frac{1}{63}$.

For $\frac{1}{4}\sin^3\theta\cos^3\theta\cos3\phi=\frac{1}{4}\mu^3\sin^3\theta\cos3\phi$

$$\mu^3=\frac{1}{6.5.4}\,\frac{d^3(\mu^6)}{(d\mu)^3}.\qquad \mu^6=\tfrac{16}{231}P_6(\mu)+\tfrac{24}{77}P_4(\mu)+\tfrac{10}{21}P_2(\mu)+\tfrac{1}{7}P_0(\mu).$$

$$\frac{d^3(\mu^6)}{(d\mu)^3}=\tfrac{16}{231}\cdot\frac{d^3P_6(\mu)}{(d\mu)^3}+\tfrac{24}{77}\cdot\frac{d^3P_4(\mu)}{(d\mu)^3}.\qquad\therefore\ \mu^3=\tfrac{2}{3465}\cdot\frac{d^3P_6(\mu)}{(d\mu)^3}+\tfrac{1}{385}\cdot\frac{d^3P_4(\mu)}{(d\mu)^3}.$$

$$\therefore\ \tfrac{1}{4}\sin^3\theta\cos^3\theta\cos\phi=-[\tfrac{2}{693}P_6{}'(\mu)-\tfrac{1}{770}P_4{}'(\mu)-\tfrac{1}{63}P_2{}'(\mu)]\cos\phi.$$

$$\tfrac{1}{4}\sin^3\theta\cos^3\theta\cos3\phi=-[\tfrac{1}{6930}P_6^3(\mu)+\tfrac{1}{1540}P_4^3(\mu)]\cos3\phi.$$

Now $P_m{}^n(\mu)=\sin^n\theta\,\dfrac{d^nP_m(\mu)}{(d\mu)^n}=(1-\mu^2)^{\frac{1}{2}n}\dfrac{d^nP_m(\mu)}{(d\mu)^n}$.

$P_6(\mu)=\frac{1}{16}(231\mu^6-315\mu^4+105\mu^2-5)$.

$P_4(\mu)=\frac{1}{8}(35\mu^4-30\mu^2+3$. $\quad P_2(\mu)=\frac{1}{2}(3u^2-1)$.

$$\therefore \sin^3\theta\cos^3\theta\sin^2\phi\cos\phi=-\tfrac{1}{44}(11\mu^3-3\mu)(1-\mu^2)^{\frac{3}{2}}\cos3\phi$$

$$-\tfrac{3}{44}\mu(1-\mu^2)^{\frac{3}{2}}\cos3\phi-\tfrac{1}{132}(33\mu^5-30\mu^3+5\mu)\sqrt{(1-\mu^2)}\cos\phi$$

$$+\tfrac{1}{308}(7\mu^3-3\mu)\sqrt{(1-\mu^2)}\cos\phi+\tfrac{1}{21}\mu\sqrt{(1-\mu^2)}\cos\phi.$$

PROBLEMS FOR SOLUTION.

ARITHMETIC.

162. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A trolley road is built between two towns, and it is found that the gross annual receipts amount to 20% of the original cost; the annual cost of repairs is 2% of the original cost; and the working expenses is \$3000 in addition to 20% of the net receipts. After a year a second road is built at the same cost as the first and it is found that the gross receipts and working expenses per year are doubled, while the cost of repairs for the new road is 1% of cost. If the net receipts for both roads is \$72,500, find the cost of each road, and the net receipts the first year.

163. Proposed by CHRISTIAN HORNUNG, A.M., Professor of Mathematics, Heidelberg University, Tiffin, O.

Three Dutchmen and their wives went to market to buy hogs. The names of the men were Hans, Klaus and Hendricks, and of the women, Gertrude, Anna and Katrine; but it was not known which was the wife of each man. They each bought as many hogs as each man or woman paid shillings for each hog, and each man spent three guineas more than his wife. Hendricks bought 23 hogs more than Gertrude, and Klaus bought 11 more than Katrine. What was the name of each man's wife?

ALGEBRA.

166. Proposed by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

Solve

$$ax+by=2zx\ldots.[1].$$
$$cy+dz=2xy\ldots.[2].$$
$$ez+fx=2yz\ldots.[3].$$

167. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A weight of m pounds falls and is broken into n pieces after which it is found that all weights, in pounds, from 1 to m can be weighed. Find the weight of each piece. Apply when $m=121$, $n=5$.

CALCULUS.

156. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find the volume common to the two solids $x^2+y^2+z^2=a^2$ and $xz^2=(a-x)(x^2+y^2)$.

157. Proposed by L. C. WALKER, A. M., Petaluma High School, Petaluma. Cal.

Two equal ellipses are tangent to each other at the vertices of the major axes. If one of them be rolled on the other, find (1) the equation and area of the curve described by the vertex, and (2) by the center.

MECHANICS.

146. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics. The Temple College, Philadelphia, Pa.

A diffraction grating, with lines .05 mm. apart is held in front of a Bunsen's burner in which common salt is volatilized, and, when viewed through a telescope it is found that the angular distances of the first, second, third, fourth, fifth, and sixth bright bands from the central one are respectively $41'$, $1°21'$, $2°2'$, $2°42'$, $3°23'$ and $4°3'$. Required the wave length of sodium light.

147. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud. England.

A particle mass m is attached to one end of a string, the other end of which is fixed. It is projected horizontally with such a velocity that it would rise to a position in which the string would be horizontal. But on its upward path it meets an inelastic particle mass m' and the height to which it rises is diminished by $1/p$th of what it would have risen. Find m', and the tensions of the string just after collision and at the greatest height of the particle.

DIOPHANTINE ANALYSIS.

106. Proposed by L. C. WALKER, A. M., Petaluma High School, Petaluma, Cal.

There is a series of rational triangles whose sides have a common difference of unity. Calling the one whose sides are 3, 4, 5 the first triangle, find the sides of the next five triangles, and a general expression for the sides of the nth triangle.

107. Proposed by L. C. WALKER, A. M., Petaluma High School, Petaluma, Cal.

Required the least three positive integral numbers such that the sum of all three of them, and the sum of every two of them shall be a square number.

NOTE.—Problem 105, Diophantine Analysis, May number, should read as follows:
Prove that every factor of $a^{2^m}+b^{2^m}$ is of the form $1(\text{mod}.2^{m+1})$ where a and b are prime to each other.

AVERAGE AND PROBABILITY.

131. Proposed by LON C. WALKER, A. M., Petaluma High School, Petaluma, Cal.

A sphere is described with its center within a given sphere, and its surface intersecting the surface of the given sphere. The average volume common to both spheres is 10/21 of the volume of the given sphere..

132. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

n points are taken at random on the circumference of a given circle. Prove that the chance of the center of the circle lying within the polygon formed by joining these points is $1-(1/2^{n-2})$.

MISCELLANEOUS.

128. Proposed by J. E. SANDERS, Hackney, Ohio.

The sides of a trapezium are $a=29$, $b=32$, $c=40$, and $d=36$. If c is opposite a, and the diagonals equal, what is the length of either diagonal?

129. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

How high above the surface of the earth must an observer be elevated at the latitude $\phi(=39^\circ\ 19')$, the declination of the sun being $\delta(=23^\circ\ 27')$, in order to see the sun at midnight?

NOTES.

Professor A. G. Greenhill was awarded by the London Mathematical Society its De Morgan Medal for 1902.

Professor W. H. Metzler, of Syracuse University, has been made Fellow of the Royal Society of Edinburgh.

Professor L. L. Locke has been elected Professor of Mathematics in Adelphi College, Brooklyn, New York.

Professor I. L. Fuchs, Professor of Mathematics in the University of Berlin since 1884, and of late editor of *Crelle's Journal*, died April 26th, at the age of sixty-eight years.

Dr. Charles W. M. Black, Instructor in Mathematics in the University of Oregon, and a contributor to the MONTHLY during the first two or three years of its publication, died August 11, at La Grande, Oregon.

Professors Ormond Stone, of the University of Virginia, E. H. Moore, of the University of Chicago, and Frank Morley, of Johns Hopkins University, have been appointed by the executive committee of the Carnegie Institution, as advisors in relation to original research in mathematics.

On July 6th occurred the death of William Lee Harvey, of Portland, Me. Mr. Harvey was born at Maxfield, Me., November 18, 1825. He was born and raised on a farm in the backwoods of Maine, and in his early years had only the advantages of the district schools of that day. He managed to spend a few terms in an academy and thus prepared himself for teaching in the common schools. While in school he acquired a taste for mathematics, and studied and

pretty thoroughly mastered algebra, geometry, trigonometry, and the calculus, without a teacher, using as texts Bonnycastle, Hutton, Young, and Peirce. As a problem-solver, Mr. Harvey ranked high, as many of his solutions of difficult problems proposed in the various mathematical journals of the country will show. He has in his library some rare works on mathematics, all of which are offered for sale. Persons interested in securing rare mathematical works should write to Mrs. Harvey, Portland, Me.

Since the last issue of the MONTHLY one of its loyal friends and supporters has been removed from among the great body of American mathematicians. On July 8th, occurred the death of Professor John D. Runkle, Professor of Mathematics in the Massachusetts Institute of Technology. Professor Runkle was a substantial friend of the MONTHLY from the beginning, and we shall greatly miss him. In the last letter we received from him, among other things, he said, "Please find enclosed my check for $5 in payment of my subscription to the MONTHLY for the current year. I am glad to make this small voluntary contribution to help you in your difficult work, for I know, by experience, how hard it is to maintain the publication of a mathematical journal in this country. You are doing a good work." Professor Runkle's experience with mathematical journalism was with the *Mathematical Monthly* which was founded by him in 1858 and which he edited and published for three years, at the end of which time it was discontinued for lack of proper support. We hope in a future number to publish a biographical sketch of Professor Runkle.

BOOKS AND PERIODICALS.

The Foundations of Geometry. By David Hilbert, Ph. D., Professor of Mathematics, University of Göttingen. Authorized Translation by E. J. Townsend, Ph. D., University of Illinois. 8vo. Cloth, 132 pages. Price, $1.00 Chicago: The Open Court Publishing Co.

In this work Professor Hilbert attempts "to choose for geometry a *simple* and *complete* set of independent axioms and to deduce from these the most important geometrical theorems in such a manner as to bring out as clearly as possible the significance of the different groups of axioms and the scope of the conclusions to be derived from the individual axioms."

He begins by considering *three* systems of things which he calls *points*, *straight lines*, and *planes*, and sets up a system of axioms connecting these things in their mutual relations. The axioms he arranges in five groups. These are I. 1—7. Axioms of *connection*; II. 1—5. Axioms of *order*; III. Axioms of *parallels* (Euclid's axiom); IV. 1—6. Axioms of *congruence*; V. Axiom of *continuity* (Archimedes' axiom). He then discusses the relations of these axioms to one another and also the bearing of each upon the logical development of Euclidean Geometry. With these axioms, Professor Hilbert arrives at many important results. The dependence of some of Professor Hilbert's axioms on others has been referred to in a previous issue of the MONTHLY. The work is not only of great mathematical value, but it is also of great importance from the pedagogical standpoint to the teacher of mathematics. Every teacher of geometry should carefully study this book. B. F. F.

A Philosophical Essay on Probabilities. By Pierre Simon, Marquis de Laplace. Translated from the Sixth French Edition by Frederick Wilson Truscott, Ph. D., Professor of Germanic Languages in the West Virginia University, and Frederick Lincoln Emory, M. E., Professor of Mechanics and Applied Mathematics in the West Virginia University. 8vo. Cloth, iv+196 pages. Price, $2.00 New York: John Wiley & Sons.

In this essay are set forth in a very simple way and without the aid of analysis, the principles and general results of the theory of probability. As the principles of probability obtain in the most important problems of life, this book will be of interest to the student of the humanities as well as to the student of mathematics. This is certainly the most elementary exposition of the theory that has ever been written. The fact that Laplace is the great authority on the subject from the analytical point of view gives this essay double value. B. F. F.

An Introduction to Celestial Mechanics. By F. R. Moulton, Ph. D., Instructor in Astronomy in the University of Chicago. Large 8vo. Cloth, xv+384 pages. Price, $3.50. New York: The Macmillan Co.

The author says, in the preface, "the aim has been to prepare such a book that one who has had the necessary mathematical training may obtain from it in a relatively short time and by the easiest steps a sufficiently broad and just view of the whole subject to enable him to stop with much of real value in his possession, or to pursue to the best advantage any particular portion he may choose." This is certainly an ambitious aim, but we believe that the author has well accomplished his aim. The work begins with a treatment of simple mechanical principles and solutions of problems in which the principles are used. From these elementary considerations the student is *gradually* led to the consideration of the most difficult problems in Astronomy. Logical sequence of the various subjects and the relative prominence which their scientific and educational importance deserve, have been maintained throughout the book. Many problems have been solved to illustrate the various principles established. The Introduction to the Problem of Three Bodies is very explicit. The work most successfully bridges the gap between the somewhat Popular or Descriptive treatment of Astronomy on the one hand, and the Exhaustive Mathematical treatment on the other. The work is desined to do great good from both the mathematical and astronomical standpoint. Both the author and the publishers are to be congratulated on presenting this valuable work to the educational public. B. F. F.

Annals of Mathematics. Published Quarterly under the Auspices of Harvard University. Price, $2.00 per year.

No. 4, of Vol. 3 (July number) contains the following articles: Note on a Twisted Curve with an Involution of Pairs of Points in a Plane, by Prof. H. S. White; On Some Curves Connected with a System of Similar Conics, by Prof. R. E. Allardice; Note on Multiply Perfect Numbers, by Jacob Westlund; A Mechanical Construction of Confocal Conics, by Wm. R. Ransom; On Sophus Lie's Representation of Imaginaries in Plane Geometry, by Prof. Percy F. Smith; Note on the Group of Isomorphism of a Group of Order p^m, by Dr. G. A. Miller. B. F. F.

The American Journal of Mathematics. Published Quarterly under the Auspices of Johns Hopkins University. Price, $5.00 per year.

No. 3, Vol. XXIV, contains the following articles: Die Typen der linearen Complexe elliptische Curvenim R^r, Von S. Kantor; Generalization of the Differentiation Process, by Robert E. Moritz; Simple Pairs of Parallel W-Surfaces, by Henry D. Thompson. B. F. F.

Arithmetica Razionale ad Uso dei Ginnasi. Autore Roldolfo Bettazzi, Prof. Nella R. Accademia Militare e Nel R. Liceo Cavour. Prezzo, L. 2,00 Torino.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. IX. OCTOBER, 1902. NO. 10.

THE EXPRESSION OF THE nth POWER OF A NUMBER IN TERMS OF THE nth POWERS OF OTHER NUMBERS, n BEING ANY INTEGER; AND THE DEDUCTION OF SOME INTERESTING PROPERTIES OF PRIME NUMBERS.

By J. W. NICHOLSON, A. M., LL. D., Professor of Mathematics in the Louisiana State University.

[Concluded from August-September Number.]

Making $m=7$, we have

$$8^n=6^{n+1}-\tfrac{7}{2}.4^{n+1}+7.2^{n+1}=6.6^n-14.4^n+14.2^n....(37),$$

where $n=5$, 3, or 1.

When m is even and n is odd, or vice versa, (34) becomes

$$(m+1)^n=(m+1)(m-1)^n-\frac{(m+1)(m)}{2}(m-3)^n$$

$$+\frac{(m+1)\,(m)(m-1)}{6}(m-5)^5-\text{ etc., } ....(38).$$

COR. V. In (21) m may be made equal to n provided that $n!\ a_1a_2....a_n$ be added to the second member.

Suppose these changes to be made and denote the result by....(39).

EXAMPLES.

Making $n=3$, in (39), we have

$$(c+a_1+a_2+a_3)^3=(c+a_1+a_2)^3+(c+a_1+a_3)^3+(c+a_2+a_3)^3$$
$$-(c+a_1)^3-(c+a_2)^3-(c+a_3)^3+c^3+1.2.3.a_1a_2a_3 \ldots .(40).$$

In (39), by making $a_1=a_2=\ldots .a_n$, and transposing, we have

$$n!\ a_1{}^n=(c+na_1)^n-n[c+(n-1)a_1]^n+\frac{n(n-1)}{2}[c+(n-2)a_1]^n-\ldots .$$
$$+(-1)^n c^n \ldots .(41).$$

Making $a_1=1$, we have

$$n!=(c+n)^n-n(c+n-1)^n+\ldots .-(-1)^n n(c+1)^n+(-1)^n c^n \ldots .(42).$$

Now making $c=0$, we have

$$n!=n^n-n(n-1)^n+\frac{n(n-1)}{2}(n-2)^n-\ldots .-(-1)^n n(1)^n \ldots .(43).$$

Thus, for $n=4$, we have

$$1.2.3.4=4^4-4.3^4+6.2^4-4.1^4 \ldots .(44).$$

For $n=6$ and $c=5$, (42) becomes

$$1.2.3.4.5.6=11^6-6.10^6+15.9^6-20.8^6+15.7^6-6.6^6+5^6 \ldots .(45).$$

SOME INTERESTING PROPERTIES OF PRIME NUMBERS.

In the following $m+1$ is supposed to be a prime number, and m', m'', etc. represent integers.

Since each of the binomial coefficients in (25), $+1$ or -1, is divisible by $m+1$), (25) may be written

$$(c+ma_1)^n+[c+(m-1)a_1]^n+\ldots .(c+a_1)^n+c^n=m'(m+1) \ldots .(46).$$

That is, the first member, in which c and a_1 may be any integers and n any integer less than m, is exactly divisible by the prime number $m+1$.

Making $c=0$ and $a_1=1$, (46) becomes

$$(m)^n+(m-1)^n+(m-2)^n+\ldots .2^n+1=m''(m+1) \ldots .(47).$$

In (46), writing $2m$ for m, making $a_1=1$, $c=-m$, and supposing n an even number, we have

$$(m)^n+(m-1)^n+(m-2)^n+\ldots.2^n+1^n=m'(2m+1)\ldots.(48),$$

where $2m+1$ is prime, and where n is any even number less than $2m$.

Thus, 17 will exactly divide

$$8^{2x}+7^{2x}+6^{2x}+5^{2x}+4^{2x}+3^{2x}+2^{2x}+1,$$

when $x=1, 2, 3, 4, 5, 6$, or 7.

The converse of either of the preceding properties is not always true; we will now deduce some properties which belong exclusively to prime numbers.

According to Wilson's theorem, m' being an integer,

$$1+n!=m'(n+1)\ldots.(49)$$

only when $n+1$ is a prime number.

When $n+1$ is a prime number, (42) may be written

$$n!=(c+n)^n+(c+n-1)^n+(c+n-2)^n+\ldots.c^n-m'(n+1)\ldots.(50).$$

Adding 1 to both sides and we readily obtain

$$(c+n)^n+(c+n-1)^n+(c+n-2)^n+\ldots.c^n+1=m'(n+1)\ldots.(51),$$

which is true only when $n+1$ is a prime number.

Making $c=0$, and we have

$$n^n+(n-1)^n+(n-2)^n+\ldots.1+1=m'(n+1)\ldots.(52).$$

That is, S_n+1 is exactly divisible by $n+1$, when the latter is a prime number, and only when it is prime, where $S_n=1+2^n+3^n\ldots.n^n$.

In (51), by writing $2n$ for n, and $-n$ for c, and reducing, we have

$$2[n^{2n}+(n-1)^{2n}+\ldots.2^{2n}+1]+1=m'(2n+1)\ldots.(53),$$

which is also true only when $2n+1$ is a prime number.

Subtract $2n+1$ from both members of (53), and divide by $2n+1^x$, we have

$$\{[n^{2n}-1]+[n-1)^{2n}-1]+\ldots.[2^{2n}-1]\}\div(2n+1)=m''.$$

That is, if each of the quantities in the [ ] is exactly divisible by $2n+1$, then $2n+1$ is a prime number. This may be considered a generalization of Fermat's Theorem.

SOME MODERN METHODS AND PRINCIPLES OF GEOMETRY.*

By PROFESSOR HEINRICH MASCHKE, Ph. D.

It might be said of the most important parts of recent geometry that one conception dominates everywhere: that is the conception of the *group.* Suppose we are given a set of operations of any kind, which I call S_1, S_2, S_3, S_4, ... , finite or infinite in number, a set of operations which are defined by some law. Take now one of the operations, say S_i, apply it first, and after that has been done apply in succession another of the operations, S_k. If now it is so that the combined operation S_iS_k, which is obtained by applying first S_i and afterwards S_k, is again an operation *in the original set;* and if this is so for any two operations of the set, then the set forms a *group.* Let me give you an example. Think of a sphere with center fixed, and define a set of operations by all the possible rotations of the sphere about its center. That is an infinite number of operations. These operations, I say, form a group. Revolve the sphere first about a certain diameter through a certain angle. This is one of the operations of our set. After that has been done, take another axis and revolve the sphere about this second axis through a certain angle. Then it can be proved that the combined effect of these two rotations is equivalent to a single rotation about a certain axis and through a certain angle. The effect produced by two operations of the set applied in succession is the same as the effect of another operation contained in the set. Therefore, all these rotations form a group. The number of operations in this group is infinite.

Suppose now we have a triangle with sides of two, three, and four feet in length. Whether we make an investigation about this triangle here in this room in Ryerson Laboratory, or over in Cobb Hall, say, the result is the same. This means that in geometry our investigations are independent of the location of our figures in space. In other words, if I make a certain investigation of a certain triangle and then move that triangle to some other place in space, I do not change anything of the character of the theorem. Now instead of saying that we will move our figure from one place to another, I will rather say that we move the whole of space by that same amount which will bring this figure into coincidence with the other figure; and so then the following statement will be clear: our geometrical theorems are not changed when we submit the whole of space to a certain motion. The truth of our geometrical theorems is independent of the motion of space. If we consider all the possible motions of the whole of space, then these motions form a group, because the application of two motions in succession is equivalent to one single motion. Every motion can be considered as a transformation in the following sense: Suppose we take a point, and fix it by some means, say by its coördinates x, y, z; then by any motion of the space the point

*Read at the fifteenth educational conference of the academies and high schools affiliating or co-operating with the University of Chicago. With some modifications, the first part of this paper appeared in *The School Review*, Jauuary, 1902.

(x, y, z) goes into another point (say x', y', z'); and so every point of space is *transformed* into some other point. What we consider is this transformation, this connection between the points in the old position and the new position. Now, whenever the notion of a group comes in there is always the question of what remains invariant under such a group. If we subject the space to all possible motions, the most important invariant is the distance between two points. Take any two points, A and B; however you may move your space by translation, or rotation, or whatever you like, the distance between A and B remains always the same: it is an *invariant*. Also the angle between any two lines is invariant under this group of all possible motions in space. Of course these are not the only invariants. Indeed, every geometrical property—the theorem that the three perpendiculars at the middle points of the three sides of a triangle meet in a point, and all similar theorems—is independent of the accidental location of the triangle in space; all these theorems have an invariant character.

Let us go a step further. Take some scalene triangle, ABC, and consider the symmetrical triangle $A'B'C'$—all sides and angles equal respectively, but lying in the opposite direction. It is possible to make them lie one on the other by a certain motion. Take the line of symmetry, and revolve the plane of the first triangle about this line; then this triangle will cover the other one. But such a motion is not possible if you allow only motion in the plane. Let us say the triangle $A'B'C'$ is obtained from ABC by a *reflection* on their line of symmetry. In space, take a certain plane and reflect our figures on this plane. An irregular tetrahedron goes by such a reflection into another precisely equal to the first; but it is not possible by any *motion* in space to bring the two tetrahedrons into coincidence with each other. It is like the difference between the right and left hands. It would be possible to bring them together by mere motion if we could go into a space of four dimensions,* but it is not possible in space of three dimensions; just as in the case of the two triangles, where it is not possible to bring them into coincidence by motion in a plane, but only by motion in space of three dimensions.

But now I say in our geometrical investigations it does not make any difference whether we consider a certain figure or a figure which is deduced from the first one by such a reflection.

Let us consider all possible reflections in space on all possible planes. The question is, do they form a group? The answer is, no, because one reflection on one plane changes a given tetrahedron into a symmetrical tetrahedron, and any other reflection on a second plane changes the second tetrahedron into its symmetrical tetrahedron, which is equal and equally directed to the first, so that by two successive reflections we do not get again a reflection, but something which is equivalent to a motion. If, however, we join to all possible motions of space all possible reflections, this totality again forms a group, because no matter how you combine any motions and reflections, you always get either a motion or a reflection: that is to say, you get again an operation of the set. What is invar-

*As to the space of four dimensions, see the explanations given at the end of the paper.

iant under this group? The distance between any two points, the angle between any two lines, and in the third place, every elementary geomotrical theorem.

Again let us go a step further. Suppose we investigate a triangle with sides respectively two, three, and four feet in length. A teacher in Paris does not say *feet*, but twenty, thirty, forty centimeters—a different size; but the theorems which he deduces from his triangle are the same as the theorems which we deduce. In other words, for our elementary geometrical theorems the size is immaterial. We allow then an expansion or reduction in size, everything remaining similar, of course. To fix the ideas let us define such an expansion or reduction in this way: Take a fixed point, and join it to all points in space by lines called radii vectores, and change every radius vector, without changing the angles, in the ratio $l:n$; the effect will be the expansion or reduction of the whole of space in size. Now let us join to all operations of our group containing all possible motions and reflections all these expansions and reductions; the combined operations again form a group, and this group has been called by Klein the *principal group* of geometry. Our geometrical theorems then remain true under this principal group: that is to say, they remain true if we apply any one of the operations of this principal group—any motion and reflection, or any expansion or reduction in size.

If we ask about invariants, we see at once that under this group distance is not invariant. But the ratio of two distances is invariant; it remains, of course, invariant for every motion and every reflection, and also for every expansion or reduction. The angle between two lines is also an invariant under the principal group. With this conception of the principal group we might give the following definition of the subject-matter of elementary geometry. We might say it is the establishment and deduction of geometrical properties which remain unchanged under this principal group.

Let us now extend this group by joining other operations. We then come right into the midst of modern geometry. Take any plane figure in space, on the board, for instance, and now take a point not in the plane of the board, and join this point to all the points of your figure: let the point be your eye, say, and let the straight lines be the lines on which you look upon the different points. If now you take a plane and place that plane in any position between the point and the board, there results what is called a projection of the figure on the board on this new plane. Let A be a point in the plane of the board, and O your center of projection, and let the corresponding point in the second plane be A', the point of intersection of the plane with OA. Thus every point A goes into a definite point A'. How does this figure in the second plane differ from the figure in the first plane? Is the distance between two points preserved? Certainly not. Is the ratio of the distances of two points preserved? Certainly not in general. If you have the points A and B, and C in the middle, and project from the point O, the point C' will *not* be in the middle of $A'B'$, unless the two planes are parallel. The angles between any two lines are also changed. But there is another thing which remains invariant—the ratio of two ratios. Take the line AB and divide it at C and D. Then

$A \qquad C \qquad D \qquad B \qquad \dfrac{CA}{CB} : \dfrac{DA}{DB}$

is invariant under this projection. This is called the *cross-ratio* or *anharmonic-ratio* between these points. This projection, however, might be considered as a transformation of the plane. Take the second plane and place it on the first plane; then you have on the first plane a certain point A and its corresponding point A', B and its corresponding point B'; whence you have a transformation of the different points on that plane.

A similar transformation is possible in space; only to make that projection we have to take a point outside of space; that is, a point in the fourth dimension somewhere. From that point we project every point of our space into another three-dimensional space, and then bring that second space into coincidence with the first. Then you have the same relation as before—for every point A a new point A'.

Analytically this transformation is much simpler. It can be shown that the coördinates x', y', z' of the new points A' are rational linear functions of the coördinates x, y, z of the old points A. From these formulas it follows at once that all these transformations (they are called *projections* in the plane and *collineations* in space) form a *group.*

Apply to the x', etc., a collineation, and you get x'', etc., in terms of x, y, z, a formula of the same kind. And every formula of that kind gives a collineation. Therefore the totality of all collineations in space form a group. This group contains the principal group, because every motion, every reflection, and every expansion or reduction can always be expressed by a formula of the above kind. This is the *group of projective geometry.*

Here the distance is not any longer invariant, nor is the angle, nor is the ratio between two lines; but the cross-ratio is an invariant, indeed the most important one of this group of projective geometry. The subject matter of projective geometry is then the study of geometrical theorems which remain unchanged under this group.

There are many other possible transformations of space, and each is defined by a certain group. I mention the Cremona transformations, in which the coördinates of the new points are no longer linear, but higher rational functions of the old, and the old of the new. These transformations also form a group, and that group contains all the groups which we had before. Another very general transformation is the transformation which underlies the so-called *analysis-situs*—the investigation of all those geometrical properties which remain unchanged for every continuous *deformation.* By that I mean any deformation which is such that two points which are very near together remain very near together; such a transformation as is made by squeezing a rubber ball in your hand. This transformation is so general, one might think, that by this process we could change any figure into almost any other figure. But by squeezing a ring you can never make a sphere, and conversely, by that process of deforma-

tion you can never get a ring from a sphere. There are also several invariants under this transformation—the most important of which is the so-called *genus*.

There is another principle of modern geometry which I wish to point out in a few words. I have mentioned occasionally the *fourth dimension*. Now the new principle referred to is the free use of any number of dimensions in geometry. Since we are three dimensional beings, it is utterly impossible for us to see in our imagination any space of higher than three dimensions. The study of higher spaces is therefore, and can only be, purely analytical. We might also treat analytic geometry of three dimensions in a purely analytical way, leaving aside all geometrical notions. In this sense analytic geometry of three dimensions is simply the study of functions of three independent variables x, y, z. Likewise, analytic geometry of four dimensions is the study of functions of four independent variables x, y, z, w. But in this study we might borrow the phraseology from analytic geometry of three dimensions. We might talk of a plane, of a line, a point, a three-dimensional space in the space of four dimensions, meaning by these terms certain linear equations or systems of equations in x, y, z, w. One linear equation would represent a three-dimensional space; for instance, $w=0$ would represent the ordinary space of three dimensions. Two linear equations in x, y, z, w would represent a plane; etc. Reasoning by analogy from three-dimensional space will help us then considerably in our analytic study in four dimensions.

In a certain way, however, a direct geometrical insight into spaces of higher dimensions is possible. When we consider our ordinary space as consisting not—as we are accustomed to—of points as elements, but of straight lines, then it becomes at once a space of four dimensions, because a straight line is determined by four independent coördinates. Again, taking other simple figurations as elements of space, for instance, the sphere, the circle, or the general surface of the second order, we might endow our ordinary space with any number of dimensions we please.

In geometry of three dimensions there are only five *regular* bodies: the tetrahedron, the hexahedron, the octahedron, the dodekahedron, and the ikosahedron. If we wish to represent these regular figures of space in the plane, we take a plane and a point outside, and project on the plane the regular hexahedron, for example. In general, several of the projected edges will meet. But that can be easily avoided in the following way: Place the body under consideration on the plane, and take as point of projection a point above the middle point of one of the faces and not far from it, in such a way that the upper face is so projected that it includes all the other faces. Then no two projected edges meet. For instance, Fig. 1 shows the projection of the regular hexahedron.

Let us do the same thing in a higher space. Take the space of four dimensions. It can be shown that in this space there are six regular bodies. Our imagination fails of course to see them, but we can see the projections of these bodies into our space of three dimensions. As the center of projection, we take

a point in the space of four dimensions chosen so that no meeting of the various lines occur.

A body of four dimensions is bounded by what corresponds to faces in the body of three dimensions —*i. e.*, by a certain number of bodies of three dimensions, in such a way that all these different bodies lie in different spaces; and every one of these is bounded by planes, every plane by edges, and every edge by vertices.

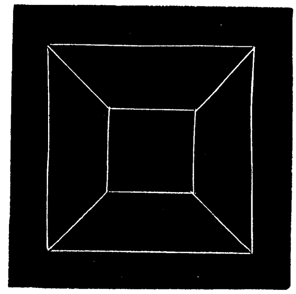

Fig. 1.

In Fig. 2 is given a perspective view of the projection of the so-called 8-cell, one of the regular bodies in four-dimensional space. We observe in the figure eight hexahedrons (counting also the one which includes all the others); these are the projections of the three-dimensional bodies (cells) which bound the four-dimensional body.

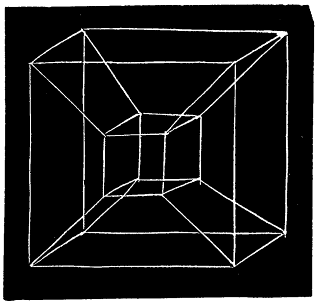

Fig. 2.

In the lecture itself, a set of wire models belonging to the mathematical department of the University of Chicago was shown to illustrate the projections into space of three dimensions of all six regular four-dimensional bodies.

The University of Chicago, October, 1902.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

148. Proposed by R. D. BOHANNAN, Ph. D., Professor of Mathematics, Ohio State University, Columbus, O.

If $\frac{x}{a+\alpha}+\frac{y}{b+\beta}+\frac{z}{c+\gamma}=1$, $\frac{x}{a+\beta}+\frac{y}{b+\beta}+\frac{z}{c+\beta}=1$, $\frac{x}{a+\gamma}+\frac{y}{b+\gamma}+\frac{z}{c+\gamma}=1$, show, without solving, that $x+y+z=a+\alpha+b+\beta+c+\gamma$.

Solution by JAMES McMAHON, A. M., Professor of Mathematics, Cornell University, Ithaca, N. Y.

There is probably a misprint in the first equation. It should be

$$\frac{x}{a+\alpha}+\frac{y}{b+\alpha}+\frac{z}{c+\alpha}=1.$$

The form of the three given equations shows that α, β, γ are the three roots of the equation

$$\frac{x}{a+\varsigma}+\frac{y}{b+\varsigma}+\frac{z}{c+\varsigma}=1,$$

in which ς is regarded as the unknown. On clearing of fractions, and arranging in the form of a cubic equation in ς, it is seen that the sum of the three roots is $-(a+b+c)+(x+y+z)$.

Hence $\alpha+\beta+\gamma=-(a+b+c)+(x+y+z)$, and $x+y+z=a+\alpha+b+\beta+c+\gamma$.

NOTE. It may be of interest to state that if each letter be squared the result expresses the distance of any point from the origin in terms of ellipsoidal curvilinear coördinates.

156. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

$(z+x)a-(z-x)b=2yz\ldots.(1)$; $(x+y)b-(x-y)c=2xz\ldots.(2)$; $(y+z)c-(y-z)a=2xy\ldots.(3)$. Find the values of x, y, and z by the method of linear simultaneous equations.

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $x=\frac{1}{2}(b+c)u$, $y=\frac{1}{2}(a+c)v$, $z=\frac{1}{2}(a+b)w$.

$\therefore$ $(a-b)w+(b+c)u=(a+c)vw\ldots.(1)$.

$(b-c)u+(a+c)v=(a+b)uw\ldots.(2)$.

$(c-a)v+(a+b)w=(b+c)uv\ldots.(3)$.

We might eliminate v, w and get an equation of the fifth degree in u. We will, however, proceed as follows: Add (1), (2), (3), then

$$aw(2-u-v)+bu(2-v-w)+cv(2-u-w)=0.$$

This is the case when $u=v=w=0$; or $u=v=w=1$; or $u=0$, $w=v=2$; or $v=0$, $u=w=2$; $w=0$, $u=v=2$.

The first two sets of values satisfy the conditions.

$\therefore$ $x=y=z=0$; $x=\frac{1}{2}(b+c)$, $y=\frac{1}{2}(a+c)$, $z=\frac{1}{2}(a+b)$.

NOTE. This is exercise 31, page 224, Systems of Linear Simultaneous Equations, of Fisher and Schwatt's *Higher Algebra*, and has given teachers of algebra throughout the country considerable trouble. Solving the equations for a, b, and c, we readily find that

$$a=-x+y+z,$$
$$b=x-y+z, \text{ and}$$
$$c=x+y-z.$$

$\therefore$ $x=\frac{1}{2}(b+c)$, $y=\frac{1}{2}(a+c)$, $z=\frac{1}{2}(a+b)$, as one set of values for x, y, and z. EDITOR F.

Also solved by *L. C. WALKER*.

satisfied by the *four* values $\theta=\lambda$, $\theta=\mu$, $\theta=\nu$, $\theta=\rho$, in virtue of the given equations; hence it must be an identity.

To find the value of x, multiply up by $a+\theta$, and then put $a+\theta=0$; thus

$$x=\frac{(a+\lambda)(a+\mu)(a+\nu)(a+\rho)}{(a-b)(a-c)(a-d)}.$$

By symmetry, we have

$$y=\frac{(b+\lambda)(b+\mu)(b+\nu)(b+\rho)}{(b-c)(b-d)(b-a)},$$

$$z=\frac{(c+\lambda)(c+\mu)(c+\nu)(c+\rho)}{(c-d)(c-a)(c-b)},$$

and $u=\dfrac{(d+\lambda)(d+\mu)(d+\nu)(d+\rho)}{(d-a)(d-b)(d-c)}.$

Similarly solved by *G. B. M. ZERR.*

GEOMETRY.

185. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud. England.

Given the tangential equations to two conics S, S', find the tangential co-ordinates of the join of the poles of two given parallel lines with respect to S. Deduce the tangential equation of the center of S, and find that of the intersection of S and S'.

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $bc-f^2=A$, $ca-g^2=B$, $ab-h^2=C$, $gh-af=F$, $hf-bg=G$, $fg-ch=H$.

Then $S=A\lambda^2+B\mu^2+C\nu^2+2F\mu\nu+2G\nu\lambda+2H\lambda\mu$.

Similarly, $S'=A'\lambda^2+B'\mu^2+C'\nu^2+2F'\mu\nu+2G'\nu\lambda+2H'\lambda\nu$.

Let $\lambda\alpha+\mu\beta+\nu\gamma$ and $\lambda\alpha+\mu\beta+\nu\gamma+m$ be the two given parallel lines; p, q, t and p', q', t' their poles with respect to S. Then for the first line

$$ap+hq+gt=\lambda,\ hp+bq+ft=\mu,\ gp+fq+ct=\nu.$$

Solving these equations for p, q, t,

$$p=(A\lambda+H\mu+G\nu)/\triangle,\ q=(H\lambda+B\mu+F\nu)/\triangle,$$
$$t=(G\lambda+F\mu+C\nu)/\triangle,\text{ where }\triangle=abc+2fgh-af^2-bg^2-ch^2.$$

For the second line,

$$ap'+hq'+gt'+m=\lambda,\ hp'+bq'+ft'+m=\mu,\ gp'+fq'+ct'+m=\nu.$$

$\therefore\ p'=[A\lambda+H\mu+G\nu-m(A+H+G)]/\triangle$.

$q'=[H\lambda+B\mu+F\nu-m(H+B+F)]/\triangle$.

$t'=[G\lambda+F\mu+C\nu-m(G+F+C)]/\triangle$.

(p, q, t), (p', q', t') are the tangential co-ordinates of the join of the poles.

Let A', B', C' be the angles of the triangle of reference. The center is the pole of the line at infinity $\alpha\sin A'+\beta\sin B'+\gamma\sin C'=0$. The tangential co-ordinates of the center are obtained by substituting $\sin A'$, $\sin B'$, $\sin C'$ for λ, μ, ν in p, q, t and are

$$S_1=(A\sin A'+H\sin B'+G\sin C')/\triangle,$$

$$S_2=(H\sin A'+B\sin B'+F\sin C')/\triangle,$$

$$S_3=(G\sin A'+F\sin B'+C\sin C')/\triangle.$$

$\therefore$ The tangential equation of the center is $\lambda S_1+\mu S_2+\nu S_3=0$.

Write $a+ka'$ for a, $b+kb'$ for b, $c+kc'$ for c, $f+kf'$ for f, $g+kg'$ for g, $h+kh'$ for h in $A\lambda^2+B\mu^2+C\nu^2+2F\mu\nu+2G\nu\lambda+2H\lambda\mu=0$.

Then the tangential equation of the four points of intersection of S and S' is $S+k\Phi+k^2S'=0$ where k is undetermined, and

$$\Phi=(bc'+b'c-2ff')\lambda^2+(ca'+c'a-2gg')\mu^2+(ab'+a'b-2hh')\nu^2$$
$$+2(gh'+g'h-af'-a'f)\mu\nu+2(hf'+h'f-bg'-b'g)\pi\lambda$$
$$+2(fg'+f'g-ch'-c'h)\lambda\mu.$$

The condition for equal roots for k is $\Phi^2=4SS'$, which is the equation of the four points of intersection.

186. Proposed by J. R. HITT; Professor of Mathematics, Coronal Institute, San Marcos, Texas.

If two sides of a triangle and its in-circle be given in position, tne envelope of its circumcircle is a circle (*Mannheim*). [From Casey's *Sequel to Euclid*.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let vertex A be origin, sides b, c the axes. Then $x^2+2xy\cos A+y^2-bx-cy=0$ is the equation to the circumcircle. Let this equation be written

$$D-bx-cy=0\ldots.(1).$$

Since the sides b, c and the inscribed circle are fixed in position, the tangents from A to the in-circle are constant.

$\therefore\ b+c-a=$a constant$=m\ldots.(2)$.

$a=\surd(b^2+c^2-2bc\cos A)$. This in (2) gives after reduction,

$$m^2+2bc(1+\cos A)-2m(b+c)=0\ldots.(3).$$

c from (1) in (3) gives

$$2b^2x(1+\cos A)+2b[my-mx-D(1+\cos A)]+2Dm-m^2y=0.$$

The condition for equal roots of b is

$$2x(1+\cos A)(2Dm-m^2y)=[my-mx-D(1+\cos A)]^2$$

$$\text{or } [D(1+\cos A)-m(x+y)]^2=2m^2xy(1-\cos A)=4m^2xy\sin^2\tfrac{1}{2}A.$$

$$\therefore\ D(1+\cos A)-m(x+y)\pm 2m\sqrt{(xy)}\sin\tfrac{1}{2}A=0.$$

$$\therefore\ x^2+2xy\cos A+y^2-\frac{m}{2\cos^2\frac{1}{2}A}[x+y\pm 2(xy)\sin\tfrac{1}{2}A]=0.$$

$$\therefore\ x^2+2xy+y^2-4xy\sin^2\tfrac{1}{2}A-\frac{m}{2\cos^2\frac{1}{2}A}[x+y\pm 2\sqrt{(xy)}\sin\tfrac{1}{2}A]=0.$$

$$\therefore\ [x\pm 2\sqrt{(xy)}\sin\tfrac{1}{2}A+y][x\mp 2\sqrt{(xy)}\sin\tfrac{1}{2}A+y-\frac{m}{2\cos^2\frac{1}{2}A}]=0.$$

$$\therefore\ x\mp 2\sqrt{(xy)}\sin\tfrac{1}{2}A+y-\frac{m}{2\cos^2\frac{1}{2}A}=0,$$

$$\text{or } x^2+2xy\cos A+y^2+[m-4m(x+y)\cos^2\tfrac{1}{2}A]/4\cos^4\tfrac{1}{2}A=0.$$

This is the circle.

CALCULUS.

144. Proposed by G. B. M. ZERR. A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find the volume of the sphere, $x^2+y^2+z^2=2az$, (a) within the paraboloid $z=Ax^2+By^2$; (b) within the cone $z^2=Ax^2+By^2$.

Solution by the PROPOSER.

$x^2+y^2+z^2=2az\ldots.(1)$, $z=Ax^2+By^2\ldots.(2)$, $z^2=Ax^2+By^2\ldots.(3)$.

From (1), $z=a\pm\sqrt{\{a^2-x^2-y^2\}}=a\pm\sqrt{\{a^2-r^2\}}$.

From $v=\int\int zrdrd\theta$ we get

$$v=4\int_0^{\frac{1}{2}\pi}\int_0^R\sqrt{(a^2-r^2)}d\theta rdr=\tfrac{4}{3}\int_0^{\frac{1}{2}\pi}]a^3-(a^2-r^2)^{\frac{3}{2}}]d\theta.$$

(a). From (1) and (2),

$$x^2+y^2+(Ax^2+By^2)^2=2a(Ax^2+By^2).$$

$$r^2+r^4(A\cos^2\theta+B\sin^2\theta)^2=2ar^2(A\cos^2\theta+B\sin^2\theta).$$

$$\therefore r^2=\frac{2a(A\cos^2\theta+B\sin^2\theta)-1}{(A\cos^2\theta+B\sin^2\theta)^2}=R^2.$$

$$\therefore v=\frac{4a^3}{3}\int_0^{\frac{1}{2}\pi}\left[1-\left(\frac{A\cos^2\theta+B\sin^2\theta-1}{A\cos^2\theta+B\sin^2\theta}\right)^3\right]d\theta$$

$$=\frac{4a^3}{3}\int_0^{\frac{1}{2}\pi}\left[\frac{3}{A\cos^2\theta+B\sin^2\theta}-\frac{3}{(A\cos^2\theta+B\sin^2\theta)^2}+\frac{1}{(A\cos^2\theta+B\sin^2\theta)^3}\right]d\theta$$

$$=\frac{2\pi a^3}{\sqrt{AB}}\left[1-\frac{A+B}{2AB}+\frac{A^2+B^2}{8A^2B^2}\right]+\frac{\pi a^3}{6\sqrt{\{A^3B^3\}}}.$$

(*b*). From (1) and (3), $R=\dfrac{2a\sqrt{\{A\cos^2\theta+B\sin^2\theta\}}}{1+A\cos^2\theta+B\sin^2\theta}=\dfrac{2aC}{D}$.

$$\therefore v=\frac{4a^3}{3}\int_0^{\frac{1}{2}\pi}\left[1-\left(\frac{C^2-1}{D}\right)^3\right]d\theta=\frac{8a^3}{3}\int_0^{\frac{1}{2}\pi}\left(\frac{3}{D}-\frac{6}{D^2}+\frac{4}{D^3}\right)d\theta$$

$$=\frac{2\pi a^3}{\sqrt{\{(A+1)(B+1)\}}}\left[\left(\frac{A}{A+1}\right)^2+\left(\frac{B}{B+1}\right)^2\right]+\frac{4\pi a^3}{3\sqrt{\{(A+1\}^3\{B+1\}^3}}.$$

145. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find the surface bounding the volume required in problem 102.

Solution by the PROPOSER.

The surface is composed of a portion of the surface of the cone, and a portion of the surface of the paraboloid.

The equation to the cone is $x^2+y^2=c^2z^2$.

The equation to the paraboloid is $y^2+z^2=4a(a+x)$.

$S=S_c+S_p$.

$$S_c=\frac{2}{c}\int_{x_2}^{x_1}\int_0^{y_1}\sqrt{\{1+c^2\}}dxdy,\ \{\text{see 102 for limits}\}$$

$$=\frac{2}{c}\int_{x_2}^{x_1}\sqrt{\{4ac^2(a+x)-x^2\}}dx.$$

Let $x=2ac^2-2ac\sqrt{\{1+c^2\}}\cos\theta$.

$$\therefore S_c=8a^2c(1+c^2)\int_0^{\pi}\sin^2\theta d\theta=4\pi a^2c(1+c^2).$$

$$S_p=4\int_{x_2}^{x_1}\int_0^{y_1}\sqrt{\frac{a(2a+x)}{4a(a+x)-y^2}}dxdy=4\int_{x_2}^{x_1}\sqrt{\{a(2a+x)\}}\sin^{-1}\sqrt{\frac{4ac^2(a+x)-x^2}{4a(1+c^2)(a+x)}}dx$$

$$=\frac{4}{3a}\int_{x_2}^{x_1}\frac{x(2a^2+ax)^{\frac{3}{2}}}{(a+x)\sqrt{\{4ac^2(a+x)-x^2\}}}dx$$

$$=\frac{4a}{3}\int_{x_2}^{x_1}\frac{x\sqrt{\{2a^2+ax\}}dx}{(a+x)\sqrt{\{4ac^2(a+x)-x^2\}}}+\frac{4}{3}\int_{x_2}^{x_1}\frac{x\sqrt{\{2a^2+ax\}}dx}{\sqrt{\{4ac^2(a+x)-x^2\}}}.$$

Let $x=2ac^2+2ac\sqrt{\{1+c^2\}}\cos2\theta$, $1+c^2+c\sqrt{\{1+c^2\}}=b^2$, $2c\sqrt{\{1+c^2\}}/b^2=e^2$, $4c\sqrt{\{1+c^2\}}/\{2b^2-1\}=d$.

$$\therefore S_p=\frac{16a^2b\sqrt{2(b^2-1)}}{3(2b^2-1)}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1-d\sin^2\theta)\sqrt{\{1-e^2\sin^2\theta\}}}$$

$$-\frac{16a^2b\sqrt{2}(2c\sqrt{\{1+c^2\}}+b^2e^2-e^2)}{3(2b^2-1)}\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{(1-d\sin^2\theta)\sqrt{\{1-e^2\sin^2\theta\}}}$$

$$+\frac{32a^2bce^2\sqrt{\{2(1+c^2)\}}}{3(2b^2-1)}\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta d\theta}{(1-d\sin^2\theta)\sqrt{\{1-e^2\sin^2\theta\}}}$$

$$+\tfrac{16}{3}a^2b\sqrt{2(b^2-1)}\int_0^{\frac{1}{2}\pi}\sqrt{\{1-e^2\sin^2\theta\}}d\theta$$

$$-\tfrac{32}{3}a^2bc\sqrt{\{2(1+c^2)\}}\int_0^{\frac{1}{2}\pi}\sqrt{\{1-e^2\sin^2\theta\}}\sin^2\theta d\theta.$$

$$\therefore S_p=A\Pi(e,\ -d,\ \tfrac{1}{2}\pi)+\frac{B}{d}\{F(e,\ \tfrac{1}{2}\pi)-\Pi(e,\ -d,\ \tfrac{1}{2}\pi)\}$$

$$+\frac{C}{d^2e^2}\{e^2\Pi(e,\ -d,\ \tfrac{1}{2}\pi)+dE(e,\ \tfrac{1}{2}\pi)-(d+e^2)F(e,\ \tfrac{1}{2}\pi)\}+DE(e,\ \tfrac{1}{2}\pi)$$

$$+\frac{E}{3e^2}\{(1-2e^2)E(e,\ \tfrac{1}{2}\pi)-(1-e^2)F(e,\ \tfrac{1}{2}\pi)\}.$$

$$\therefore S=4\pi a^2c(1+c^2)+\left(A-\frac{B}{d}+\frac{C}{d^2}\right)\Pi(e,\ -d,\ \tfrac{1}{2}\pi)+\left(\frac{C}{de^2}+D\right.$$

$$\left.+\frac{E(1-2e^2)}{3e^2}\right)E(e,\ \tfrac{1}{2}\pi)+\left(\frac{B}{d}-\frac{C(d+e^2)}{d^2e^2}-\frac{E(1-e^2)}{3e^2}\right)F(e,\ \tfrac{1}{2}\pi).$$

MECHANICS.

139. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College. Defiance, Ohio.

A homogeneous sphere, radius $r=50$ inches, makes $m=30$ revolutions around an axis every second. The mass begins to disappear from the surface into space at a rate exactly sufficient to cause the diameter to decrease uniformly at the rate of $(1/n)$th$=1/1000$th of

a linear inch per second. At what rate per second is the angular velocity of the sphere changing the instant the diameter becomes $p=10$ inches? What is the diameter of the sphere when the rate of disappearance of matter is midway between minimum and maximum? When is the angular velocity a maximum? How does this maximum angular velocity compare with the original angular velocity? What is the diameter of the sphere when the paracentric force is (1) a maximum, and (2) a minimum?

No solution of this problem has been received.

140. Proposed by J. F. LAWRENCE, A. B., St. Louis. Mo.

A long row of particles, each mass m, is placed on a smooth horizontal table. Each is connected with the two adjacent ones by similar light elastic strings of natural length l. They receive small longitudinal disturbances such that each of them proceeds to perform a harmonic oscillation. Prove that there will be two waves of vibration in opposite directions with the same velocity, viz, $l'\sqrt{\frac{E}{ml}}\frac{q}{\pi}\sin\frac{\pi}{q}$, when l' is the average distance between two successive particles, q the number of intervals between two particles in the same phase, and E the modulus of elasticity. [*Mathematical Tripos*, 1873.]

Solution by G. B. M ZERR. A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $D=d/dt$, and the equation of motion is $y_{k+1}-2y_k+y_{k-1}=\frac{D^2}{c^2}y_k$ where $c^2=E/(ml)$. To solve this equation of differences, treat D as constant and put $y_k=Cx^k$ where C and x are two constants. Substituting and reducing, $x-2+1/x=(D/c)^2$.

$$\therefore\ \sqrt{x}-1/\sqrt{x}=\pm D/c.\quad \sqrt{x}+1/\sqrt{x}=\pm\sqrt{\{4+(D/c)^2\}}=\pm 2\sqrt{1+\left(\frac{D}{2c}\right)^2}$$

$$\therefore\ \sqrt{x}=\sqrt{1+\left(\frac{D}{2c}\right)^2}-\frac{D}{2c}=E.$$

$$\therefore\ y_k=E^{2k}f(t)+E^{-2k}F(t).$$

Let us express $f(t)$ and $F(t)$ in a series whose general term is $A\cos(2c\sin\theta t+\omega)$.

The operator E under the radical contains only even powers of D and we can write $-(2c\sin\theta)^2$ for D^2.

$$\therefore\ E\cos(2c\sin\theta t+\omega)=\cos(2c\sin\theta t+\omega-\theta).$$

$$\therefore\ E^{2k}\cos(2c\sin\theta t+\omega)=\cos(2c\sin\theta t+\omega-2k\theta).$$

$$E^{-2k}\cos(2c\sin\theta t+\omega)=\cos(2c\sin\theta t+\omega+2k\theta).$$

$$\therefore\ y_k=\Sigma A\cos(2c\sin\theta t+\omega-2k\theta)+\Sigma B\cos(2c\sin\theta t+\omega+2k\theta).$$

If we substitute in any one term of the first series $k+1$ for k and $t+T$ for t, where $T=\frac{\theta}{c\sin\theta}$, the term is unaltered..

$\therefore$ Any one term of the first series represents a wave which travels the space between one particle and the next in time T. In the same way the corresponding term of the second series represents a wave which travels in the opposite direction with the same velocity.

Now v=velocity$=l'/T=l'c\sin\theta/\theta$; but $\theta=\pi/q$ and $c=\sqrt{\frac{E}{ml}}$.

$$\therefore\ v=l'\sqrt{\frac{E}{ml}}\frac{q}{\pi}\sin\frac{\pi}{q}.$$

141. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud. England.

A simple pendulum hangs from a bicycle moving in a straight line. What deflection is produced by putting on the brake so as to exert on the machine a force equal to the nth of its weight?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College. Philadelphia, Pa.

Let v=velocity of retardation, W=weight.

Then $v=\frac{Wg}{nW}t=\frac{g}{n}t$. Let $t=1$. $\therefore\ v=g/n$.

If θ=angle of deflection of the pendulum, and l=its length, then $v^2=g^2/n^2=2gl(1-\cos\theta)$.

$$\therefore\ \cos\theta=\frac{2ln^2-g}{2ln^2}\text{ or }\theta=\cos^{-1}\left(\frac{2ln^2-g}{2ln^2}\right).$$

142. Proposed by GEORGE R. DEAN, B. Sc., Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo,

An infinite mass of liquid is bounded by the plane zx, on which are small corrugations given by $y=\phi(x)$. The velocity of the liquid at an infinite distance from the plane is parallel to x and equal to V. Prove that the velocity potential is $V_x+\frac{V}{\pi}\int_{-\infty}^{\infty}\frac{(x-\lambda)\phi(\lambda)d\lambda}{y^2+(x-\lambda)^2}$. [Bassett's *Hydrodynamics*.]

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let f=velocity potential.

$$\text{Then }\frac{d^2f}{dx^2}+\frac{d^2f}{dy^2}=\frac{d^2f}{dx^2}\left(1-\frac{1}{[\varphi'(x)]^2}\right)=0.$$

$$\therefore\ \frac{df}{dx}=\text{constant}=v\text{ when }y=\infty.\quad \therefore\ f-C=v\int dx=v\varphi_1(x).$$

When $y=0$, $f=\varphi_1(x)v$; when $y=\infty$, $\varphi(x)=x$.

$\therefore\ C=vx$. $\therefore\ f=vx+v\varphi_1(x)$.

By Fourier's series, $\varphi_1(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(\lambda)d\lambda\int_0^{\infty}\sin\beta(x-\lambda)d\beta$.

We must build up a value of $\varphi_1(x)$ satisfying the last expression.

Take $e^{-\beta y}\sin\beta x$ and $e^{-\beta y}\cos\beta x$ and multiply the first by $\cos\beta\lambda$, the second by $\sin\beta\lambda$. Subtract these and we get $e^{-\beta y}\sin\beta(x-\lambda)$.

$$\therefore\ \varphi_1(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty e^{-\beta y}\varphi(\lambda)\sin\beta(x-\lambda)d\beta d\lambda.$$

$$\text{Now }\int_0^\infty e^{-ax}\sin mx dx=\frac{m}{a^2+m^2}.$$

$$\therefore\int_0^\infty e^{-\beta y}\sin\beta(x-\lambda)d\beta=\frac{x-\lambda}{y^2+(x-\lambda)^2}.$$

$$\therefore\ \varphi_1(x)=\frac{1}{\pi}\int_{-\infty}^\infty\frac{(x-\lambda)\ \varphi(\lambda)d\lambda}{y^2+(x-\lambda)^2}.$$

$$\therefore f=vx+\frac{v}{\pi}\int_{-\infty}^\infty\frac{(x-\lambda)\ \varphi(\lambda)d\lambda}{y^2+(x-\lambda)^2}.$$

143. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Beads are fastened at equal intervals on a string placed over a smooth fixed pulley. If the original position of the string is one of symmetry, find the velocity at any moment, the pressure on the pulley, and the velocity with which the string leaves the pulley.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

In this solution the weights of the pulley and string are neglected.

Let $2l$=length of string, $2n+1$=number of beads each mass m.

Acceleration$=f=\dfrac{m[(n+x)-(n-x+1)]g}{m(2n+1)}=\dfrac{2x-1}{2n+1}g$, where x can have any value from 1 to $(n-1)$. l/n=distance between two beads.

Now $l/n=\frac{1}{2}ft_1^2=\dfrac{(2x-1)gt_1^2}{2(2n+1)}$.

$$\therefore t_1=\sqrt{\frac{2l(2n+1)}{gn(2x-1)}},\ v=ft_1=\sqrt{\frac{2gl(2x-1)}{n(2n+1)}}.$$

Let t be any time between the passing of the $(x+1)$th and $(x+2)$th particle over the pulley, then the velocity, v_1, at this time, is

$$v_1=\sum_{x=1}^{x=x}\sqrt{\frac{2gl(2x-1)}{n(2n+1)}}+\sqrt{\frac{2gl(2x+1)}{n(2n+1)}}t=\sqrt{\frac{2gl}{n(2n+1)}}[1+\sqrt{3}+\sqrt{5}+\sqrt{7}+\ldots$$

$$+\sqrt{(2x-1)}+\sqrt{(2x+1)}t].$$

The velocity with which the string leaves the pulley

$$=\sum_{x=1}^{x=n-1}\sqrt{\frac{2gl(2x-1)}{n(2n+1)}}=\sqrt{\frac{2gl}{n(2n+1)}}[1+\sqrt{3}+\sqrt{5}+....+\sqrt{(2n-3)}].$$

Pressure on pulley at time $t=\dfrac{4m^2(n+x+1)(n-x)}{m(2n+1)}=\dfrac{4m(n+x+1)(n-x)}{2n+1}$

DIOPHANTINE ANALYSIS.

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92. Proposed by **M. A. GRUBER, A. M.**, War Department, Washington, D. C.

Find the sides of integral right triangles when the difference of the legs is given.

Solution by **HON. JOSIAH H. DRUMMOND, LL. D.**, Portland, Me.

Let a be the difference in the legs, and x and $x+a$=legs. Then $2x^2+2ax+a^2=\square=$(say) $[px-a]^2=p^2x^2-2apx+a^2$. Whence, $x=\dfrac{2a[p+1]}{p^2-2}$. Take $p=2$, $x=3a$, and the sides are $3a$, $4a$, $5a$. Then in the formula $\dfrac{2[r+s]}{r+2s}$, we have $r/s=\frac{2}{1}$, then $p=\frac{3}{2}$, and $x=20a$, and the sides $20a$, $21a$, and $29a$, and so on *ad infinitum*.

Remark on Problem 94 by **HON. JOSIAH H. DRUMMOND, LL. D.**, Portland, Me.

In his solution of this question, Professor Zerr gives up his general demonstration a little prematurely. It is true that "For integral values of m, m^2+m+1 is not a square," but fractional values of m lead to integral values of a, b, and c. The value of m which makes the expression a square is $\dfrac{2p+1}{p^2-1}$ in which p may be any number except one. Take $p=2$, and $m=\frac{5}{3}$. Then $a=\frac{40}{9}$, $b=\frac{24}{9}$, and $c=\frac{15}{9}$. Any common multiple of a, b, and c makes the square root of $\sqrt{[a^2+b^2+c^2]}$ a square and as the denominator is a square, makes abc a square, as well as these particular values. So a, b, and c may be taken=40, 24, and 15, respectively. Then $abc\sqrt{[a^2+b^2+c^2]}=14400=49$, a square number. However, the question does not call for integral values, and I had solved the question as follows from the point at which Professor Zerr leaves it. Substituting the value of $m=\dfrac{2p+1}{p^2-1}$ in the values of a, b, and c, reducing to common denominator, $[p^2-1]^2$, we have $a=p[2p^2+5p+2]$, $b=p[p+2][p^2-1]$, and $c=2p^3=p^2-2p-1$, in which p may be any number except one. Hence, there is an indefinite number of rational triangles whose area is a square.

96. Proposed by **L. C. WALKER, A. M.**, Graduate Student, Leland Stanford University, Cal.

(a) Find the least three integral numbers such that the difference of every two of them shall be a square number; (b) find the least three square numbers such that the difference of every two af them shall be a square number.

Solution by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

(a). Take a=one of the numbers, any then if we take x^2+a and y^2+a =the other numbers, and two of the conditions are met, and we have only to make x^2-y^2 (the difference between the other two) a square. But x^2-y^2 is the expression for one of the sides of a right angled triangle. Hence we may take $x=p^2+q^2$ and $y=p^2-q^2$, or $2pq$ and the three numbers will be $[p^2+q^2]^2+a$ and $[2pq]^2+a$ or $[p^2-q^2]^2+a$ and a, in which a, p, and q may be any numbers. To obtain the least three take $a=1, p=2$, and $q=1$, and the numbers are 1, 10 or 17, and 26, the least being 1, 10 and 26.

(b). The solution of the second part of this problem is on page 113, April number. The problem was incorrectly numbered 92. ED.

97. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

Find a general expression for the radius of the sphere which, dropped in (or partly in) a right cone full of water, will displace the most water; the radius of the sphere, and the width, height and slant height of the cone to be rational integral numbers.

Solution by the PROPOSER.

Let ABC be a section of the cone passing through the axes, AB being the diameter of the cone, CD the axis, BC the slant height, and EB and EF radii of the sphere. Let $BD=a$, $BC=b$, and $CD=c$. Let $BE=BF=x$ and $DE=y$; then $BD=\surd[x^2-y^2]$; then $\frac{\pi[x+y]^3}{6}+\frac{\pi[x^2-y^2][x+y]}{2}$=contents of sphere within the cone which must be a maximum. Omitting constants, this reduces to $2x^3+3x^2y-y^3$=maximum....[1].

But $BC:BD::CE:FE$; or $b:a::c-y:x$, and $bx=ac-ay$....[2].

Differentiating, and reducing, $dy=-bdx/a$. Differentiating [1], substituting the value of dy and reducing, we have $2ax[x+y]-b[x^2-y^2]=0$. Dividing by $x+y$, we have $2ax-b[x-y]=0$....[3].

$x+y=0$ gives $x=-y$. Substituting this value in [1], the expression becomes zero. Hence, this value of x does not answer the conditions of the question. Substituting the value of y in [3] as found in [2], and reducing, we have

$$x=\frac{abc}{[b-a][b+2a]}.$$

As a, b, and c are sides of a right angled triangle, take $b=p^2+q^2$, $a=p^2-q^2$, and $c=2pq$. Then

$$x=\frac{p[p^2+q^2][p^2-q^2]}{q[3p^2-q^2]}.$$

For integrals, $x=p[p^2+q^2][p^2-q^2]$, $2a=2q[p^2-q^2][3p^2-q^2]$,

$b=q[p^2+q^2][3p^2-q^2]$, $c=2pq^2[3p^2-q^2]$,

in which p and q may be any integral numbers, p being greater than q.

E. g. Take $p=2$ and $q=1$. Then $x=30$, $2a=66$, $b=55$, and $c=44$.

Of course, we can take $a=2pq$, and $c=p^2-q^2$; but the resulting expression is not so simple and yet when reduced to integrals gives the same value for x, but cone thus obtained has different dimensions.

Also solved by G. B. M. Zerr, who gets slant height$=605m^2$, altitude$=484m^2$, radius of sphere $=330m^2$, and radius of cone$=363m^2$, where m is any integer.

98. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford University, Cal.

(*a*) Find the least three integral numbers such that if to the square of each the product of the other two be added, the three sums shall all be squares.

(*b*) Find the two least integral numbers such that not only each of them, but also their sum and their difference, when increased by unity, shall all be square numbers.

Solution by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

(*a*) Let x^2, m^2x^2 and nx^2 represent the required numbers. Then we have $1+mn=\square\ldots.(1)$; $m^2+n=\square\ldots.(2)$, and $m+n^2=\square\ldots.(3)$. Take $n=\frac{1}{4}-m$ $\ldots.(4)$, and (2) becomes a square, as (3) does, when we transpose (4) and get $m=\frac{1}{4}-n$. Substitute the value of n in (1) and we have

$$1+\frac{m}{4}-m^2=\square=(\text{say})\ [1-pm]^2.$$

Reducing, we have $m=\dfrac{8p+1}{4[p^2+1]}$ and $n=\frac{1}{4}-m=\dfrac{p[p-8]}{4[p^2+1]}$.

Take $x=4[p^2+1]$, $mx=8p+1$, and $nx=p[p-8]$, in which p may be any number greater than 8. Take $p=9$ and we have $x=328$, $mx=73$, and $nx=9$.

(*b*) Let x^2 and y^2 be the numbers. Then $x^2+y^2+1=\square=(\text{say})(m+1)^2$ and $x^2-y^2+1=\square=(\text{say})(m-1)^2$.

By adding and reducing, we get $x^2=m^2$ and $x=m$.

By substituting and reducing, we find $y^2=2m$. Take $m=2n^2$ and we have $x=2n^2$ and $y=2n$. Take $n=1$, and $x=2$, and $y=2$, and the numbers are 4 and 4; take $n=2$ and the numbers are 16 and 64.

Also solved by *G. B. M. ZERR*, and *L. C. WALKER*.

AVERAGE AND PROBABILITY.

Remark on Problem 109, by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.

Let $r=a\sin^2\theta$. Then the last integral in Professor Walker's solution is transformed as follows:

$$\frac{8}{9a}\int_0^a[3a^3\tan^{-1}(\frac{r}{a-r})-(3a^2+2ar-8r^2)\sqrt{(ar-r^2)}]dr,$$

$$=\frac{16a^3}{9}\int_0^{\frac{1}{2}\pi}(3\theta-3\sin\theta\cos\theta-2\sin^3\theta\cos\theta+8\sin^5\theta\cos\theta)\sin\theta\cos\theta d\theta,$$

$$=\frac{16a^3}{9}\int_0^{\frac{1}{2}\pi}(3\theta\sin\theta\cos\theta-3\sin^2\theta+\sin^4\theta+10\sin^6\theta-8\sin^8\theta)d\theta,$$

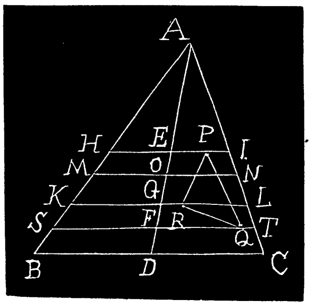
$$=\frac{16a^3}{9}(\tfrac{3}{8}\pi-\tfrac{3}{4}\pi+\tfrac{3}{16}\pi+\tfrac{25}{16}\pi-\tfrac{35}{32}\pi)=\frac{16a^3}{9}\times\frac{9\pi}{32}=\tfrac{1}{2}\pi a^3.$$

117. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A straight line is drawn at random parallel to the base of a given triangle. Three random points are then taken, one on each side of the random line and one anywhere in the triangle. Find the average area of the triangle formed by the three random points.

Solution by the PROPOSER.

Let ABC be the given triangle, MN the random line parallel to the base BC. Take P above, Q below MN, and R anywhere. Through P, Q, R draw HI, ST, KL parallel, respectively, to MN. Let $BC=2a$, $BD=DC=a$, $AD=h$, $\angle ADC=\beta$, $AE=u$, $AF=v$, $AG=w$, $EP=x$, $GR=z$, $FQ=y$, $AO=r$.



Then $EI=\dfrac{au}{h}=x'$, $GL=\dfrac{aw}{h}=z'$, $FT=\dfrac{av}{h}=y'$,

$$GJ=y-\frac{(y-x)(v-w)}{v-u}=t.$$

Area $PQR=\frac{1}{2}(t-z)(v-u)\sin\beta=A$, $t>z$; area $PQR=\frac{1}{2}(z-t)(v-u)\sin\beta=A_1$, $t<z$. The limits of r are 0 and h; of v, r and h; of u, 0 and r; of w, u and v; of x, $-x'$ and x'; of y, $-y'$ and y'; of z, $-z'$ and t, and t and z'.

$$\therefore\triangle=\frac{\int_0^h\int_r^h\int_{-y'}^{y'}\int_0^r\int_{-x'}^{x'}\int_u^v\left[\int_{-z'}^{t}Adz+\int_t^{z'}A_1dz\right]drdvdydudxdw}{\int_0^h\int_r^h\int_{-y'}^{y'}\int_0^r\int_{-x'}^{x'}\int_u^v\int_{-z'}^{z'}drdvdydudxdwdz}$$

$$=\frac{15}{a^3h^4}\int_0^h\int_r^h\int_{-y'}^{y'}\int_0^r\int_{-x'}^{x'}\int_u^v\left[\int_{-z'}^{t}Adz+\int_t^{z'}A_1dz\right]drdvdydudxdw$$

$$=\frac{15\sin\beta}{2a^3h^6}\int_0^h\int_r^h\int_{-y'}^{y'}\int_0^r\int_{-x'}^{x'}\int_u^v\left[h^2t^2+a^2w^2\right](v-u)drdvdydudxdw$$

$$=\frac{5\sin\beta}{2a^3h^6}\int_0^h\int_r^h\int_{-y'}^{y'}\int_0^r\int_{-x'}^{x'}\left[a^2(u^2+uv+v^2)+h^2(x^2+xy+y^2)\right]$$

$$\times(v-u)^2drdvdydudx$$

$$=\frac{5\sin\beta}{3a^2h^7}\int_0^h\int_r^h\int_{-y'}^{y'}\int_0^r(4a^2u^3+3a^2u^2v+3a^2uv^2+3h^2uy^2)(v-u)^2drdvdydu$$

$$=\frac{5\sin\beta}{36a^2h^7}\int_0^h\int_r^h\int_{-y'}^{y'}[3a^2r^2v^2-12a^2rv^3+18a^2v^4-12a^2r^3v+8a^2r^4$$

$$+3h^2y^2(6v^2-8rv+3r^2)]r^2drdvdy$$

$$=\frac{5a\sin\beta}{9h^8}\int_0^h\int_r^h(3r^4v^3-10r^3v^4+12r^2v^5-6r^5v^2+4r^6v)drdv$$

$$=\frac{5a\sin\beta}{36h^8}\int_0^h(3r^2h^4-8rh^5+8h^6-8r^3h^3+8r^4h^2-3r^6)r^2dr$$

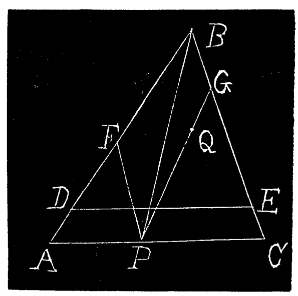
$$=\tfrac{13}{126}ah\sin\beta=\tfrac{13}{126}\text{ (area of triangle).}$$

118. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Find the mean distance between two points taken at random in an equilateral triangle.

I. Solution by L. C. WALKER, A. M., Graduate Student, Leland Stanford University, Cal.

Let ABC be any plane triangle whose sides opposite the angles A, B, C are respectively a, b, c. Let M be the required mean distance, and M_1 be the mean distance between the two points when one of the points is confined to one of the sides of the triangle, as AC. Let P be any point in the side AC, and let Q be any point in the surface of the triangle whose area$=\triangle$. Draw DE parallel to AC, and draw PF, PB, and PQG. Put $BD=u$, $AP=x$, $PQ=y$, $PF=z$, $PG=v$, $\angle BPC=\phi$, and $\angle FPC$ or $\angle GPC=\theta$. Then



$$v=\frac{c\sin A\sin(\phi+C)}{\sin\phi\sin(\theta+C)},\quad z=\frac{c\sin A\sin(\phi-A)}{\sin\phi\sin(\theta-A)},$$

$$x=\frac{c\sin(\phi-A)}{\sin\phi},\text{ and } dx=\frac{c\sin A}{\sin^2\phi}d\phi.$$

$$\text{Hence } M=\frac{\int_0^c M_1.\frac{u}{c}.\frac{u^2}{c^2}\triangle.DE.du}{\int_0^c\frac{u^2}{c^2}\triangle.DE.du}=\tfrac{4}{5}M_1$$

$$=\frac{4}{5b\triangle}\int_0^b\left[\int_0^\phi\int_0^v y^2dyd\theta+\int_\phi^\pi\int_0^z y^2dyd\theta\right]dx$$

$$=\frac{8c^3\sin^3A}{15b^2}\int_A^{\pi-C}\Big[\int_0^{\phi}\frac{\sin^3(\phi+C)}{\sin^3\phi}.\frac{d\theta}{\sin^3(\theta+C)}$$

$$+\int_{\phi}^{\pi}\frac{\sin^3(\phi-A)}{\sin^3\phi}.\frac{d\theta}{\sin^3(\theta-A)}\Big]\frac{d\phi}{\sin^2\phi}=\frac{4c^3\sin^3A}{15b^2}\int_A^{\pi-C}\Big[\frac{\cos C\sin^3(\phi+C)}{\sin^2C\sin^3\phi}$$

$$+\frac{\cos A\sin^3(\phi-A)}{\sin^2A\sin^3\phi}-\frac{\sin(\phi+C)\cos(\phi+C)}{\sin^3\phi}+\frac{\sin(\phi-A)\cos(\phi-A)}{\sin^3\phi}$$

$$+\frac{\sin^3(\phi+C)}{\sin^3\phi}\log\cot\tfrac12C+\frac{\sin^3(\phi-A)}{\sin^3\phi}\log\cot\tfrac12A-\frac{\sin^3(\phi+C)}{\sin^3\phi}\log\cot\tfrac12(\phi+C)$$

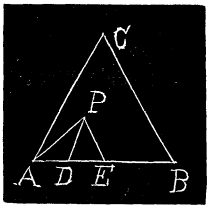
$$+\frac{\sin^3(\phi-A)}{\sin^3\phi}\log\cot\tfrac12(\phi-A)\Big]\frac{d\phi}{\sin^2\phi}=\tfrac{1}{15}\Big[\Big(\frac{a^2}{b}+\frac{b^2}{a}\Big)(\cos C+\sin^2C\log\cot\tfrac12C)$$

$$+\Big(\frac{a^2}{c}+\frac{c^2}{a}\Big)(\cos B+\sin^2B\log\cot\tfrac12B)+\Big(\frac{b^2}{c}+\frac{c^2}{b}\Big)(\cos A+\sin^2A\log\cot\tfrac12A)\Big].$$

Corollary. If $a=b=c$, then $M=\frac{a}{20}(4+3\log3)$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $\triangle$ be the required average, $\triangle_0$ the average distance from an angle to a point taken at random within the triangle. Let $AP=r$, $AB=a$, $\angle PAB=\theta$. The limits of r are 0 and $\frac12a\sqrt{3}$ cosec $(\frac23\pi-\theta)=x$; of θ, 0 and $\frac13\pi$.



$$\triangle_0=\frac{4}{a^2\sqrt{3}}\int_0^{\frac13\pi}\int_0^x r^2\,d\theta dr=\tfrac12a\int_0^{\frac13\pi}\text{cosec}^3(\tfrac23\pi-\theta)d\theta$$
$$=a(\tfrac13+\tfrac14\log3).$$

From P draw PD, PE parallel to the sides and let the area of ABC=unity. The average distance from P to any point in PDE is $u(\frac13+\frac14\log3)$ where $u=DE$. Since either of the vertices of PDE can be taken,

$$\triangle=6\int_0^a\int_0^{a-x}u(\tfrac13+\tfrac14\log3)u^2dudv=\tfrac35a(\tfrac13+\tfrac14\log3).$$

$\therefore\ \triangle=\frac35\triangle_0$.

119. Proposed by F. L. SAWYER, Mitchell, Ontario, Canada.

Two players throw three dice, the object being to throw at one cast a tr and an ace. They continue throwing in succession until one of the players wi advantage has the first player?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

The chance of winning on the first throw is $\frac{1}{216}$; the chance of first losing and second throw winning is $\frac{215}{(216)^2}$; the chance of first and second losing and third winning is $\frac{(215)^2}{(216)^3}$; the chance of the first, second and third losing and fourth winning is $\frac{(215)^3}{(216)^4}$; etc.

The chance of the first player winning is

$$\tfrac{1}{216}[1+(\tfrac{215}{216})^2+(\tfrac{215}{216})^4+(\tfrac{215}{216})^6+\ldots.]=\tfrac{216}{431}.$$

The chance of the second player winning is

$$\tfrac{1}{216}[\tfrac{215}{216}+(\tfrac{215}{216})^3+(\tfrac{215}{216})^5+(\tfrac{215}{216})^7+\ldots.]=\tfrac{215}{431}.$$

$\therefore$ The first player's chance : the second player's chance $=216:215$.

Also solved by *J. SCHEFFER*, who gets 36:35. This result is wrong. Professor Zerr's solution is correct.

Lon C. Walker sent in a solution of problem 115.

MISCELLANEOUS.

112. Proposed by L. C. WALKER, A. M., Graduate Student. Leland Stanford University, Cal.

(a) Find the area enclosed by four circles, of which two touch the x-axis, and two the y-axis at the origin.

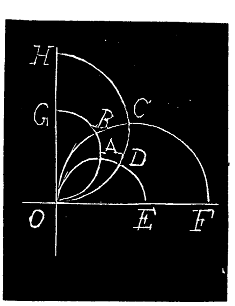
(b) Required the area enclosed by four parabolas, of which two touch the x-axis, and two the y-axis, at the origin.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and J. SCHEFFER, A. M., Hagerstown, Md.

The area $ABCD$ is the area required.

(a) Let $r=a\cos\theta\ldots.(1)$, $r=b\cos\theta\ldots.(2)$, $r=c\sin\theta\ldots.(3)$, $r=d\sin\theta\ldots.(4)$, be the equations to the circles. $a>b$, $c>d$.

Area common to (1) and (3)$=A$; area common to (1) and (4)$=B$; area common to (2) and (3)$=C$; area common to (2) and (4)$=D$. Let $\tan^{-1}(a/c)=\theta'$.



$$A=\tfrac{1}{2}a^2\int_{\theta'}^{\frac{1}{2}\pi}\cos^2\theta d\theta+\tfrac{1}{2}c^2\int_0^{\theta'}\sin^2\theta d\theta=\tfrac{1}{4}a^2\cot^{-1}(a/c)$$
$$+\tfrac{1}{4}c^2\tan^{-1}(a/c)-\tfrac{1}{4}ac.$$

$\therefore$ $B=\frac{1}{4}a^2\cot^{-1}(a/d)+\frac{1}{4}d^2\tan^{-1}(a/d)-\frac{1}{4}ad.$

$C=\frac{1}{4}b^2\cot^{-1}(b/c)+\frac{1}{4}c^2\tan^{-1}(b/c)-\frac{1}{4}bc.$

$D=\frac{1}{4}b^2\cot^{-1}(b/d)+\frac{1}{4}d^2\tan^{-1}(b/d)-\frac{1}{4}bd.$

Area $ABCD=A-C+B-D=\frac{1}{4}a^2[\cot^{-1}(a/c)$

$+\cot^{-1}(a/d)]-\frac{1}{4}b^2[\cot^{-1}(b/c)+\cot^{-1}(b/d)]+\frac{1}{4}c^2[\tan^{-1}(a/c)-\tan^{-1}(b/c)]$

$$+\tfrac{1}{4}d^2[\tan^{-1}(a/d)-\tan^{-1}(b/d)]-\tfrac{1}{4}a(c+d)+\tfrac{1}{4}b(c+d)$$

$$=\tfrac{1}{4}a^2\cot^{-1}\left(\frac{a^2-cd}{ac+ad}\right)-\tfrac{1}{4}b^2\cot^{-1}\left(\frac{b^2-cd}{bc+bd}\right)+\tfrac{1}{4}c^2\tan^{-1}\left(\frac{ac-bc}{ab+c^2}\right)$$

$$+\tfrac{1}{4}d^2\tan^{-1}\left(\frac{ad-bd}{ab+d^2}\right)-\tfrac{1}{4}(a-b)(c+d).$$

(b) Let $r=4a\cos\theta/\sin^2\theta....(1)$, $r=4b\cos\theta/\sin^2\theta....(2)$, $r=4c\sin\theta/\cos^2\theta....(3)$, $r=4d\sin\theta/\cos^2\theta....(4)$, be the equations to the parabolas; $a>b$, $c>d$.

Let $\tan^{-1}\sqrt[3]{(a/c)}=\theta'$.

$$A=8a^2\int_{\theta'}^{\frac{1}{2}\pi}\frac{\cos^2\theta}{\sin^4\theta}d\theta+8c^2\int_0^{\theta'}\frac{\sin^2\theta}{\cos^4\theta}d\theta=\tfrac{16}{3}ac.$$

Similarly, $B=\frac{16}{3}ad$, $C=\frac{26}{3}bc$, $D=\frac{16}{3}bd$.

Area $ABCD=A+B-C-D=\frac{16}{3}(a-b)(c+d)$.

113. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College Defiance, Ohio.

Deduce the Sylvestrian Reciprocant from $x^4+y^4=4x^2y^2$.

Solution by the PROPOSER.

The given equation may be written $(x^2-y^2)^2=2x^2y^2$.

$\therefore\ x^2-y^2=xy\sqrt{2}....(a)$.

Put $(y/x)=w$; then from (a), by dividing by xy, etc., we get $w-w^{-1}=-\sqrt{2}$.

$\therefore\ w=\frac{1}{2}(-\sqrt{2}\pm\sqrt{3}),=m....(b)$.

$\therefore\ y=mx$, and $dy/dx=m....(c)$.

Eliminating m from (c) by a second differentiation, we have $d^2y/dx^2=0$ $....(d)$.

Adopting Professor Sylvester's notation for reciprocants, viz: $dy/dx=t$, $d^2y/dx^2=a\ 2!$, $d^2y/dx^3=b\ 3!$, etc., we obtain from (d) the *first* pure Sylvestrian Reciprocant. All reciprocants *not* containing $t=dy/dx$ are *pure*; all others are *mixed*. The first two pure reciprocants are a and $4ac-5b^2$, and the first two mixed ones are $(1+t^2)b-2a^2t$ and $bt-a^2$. See our paper on Reciprocants published in the MONTHLY of November, 1894.

Also solved by *G. B. M. ZERR*.

114. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

When the sun's declination was 15° N. his altitude was found to be 20°, and after an hour's interval his altitude was found to be 31°. Required, the latitude of the place of observation.

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia. Pa.

Let $a=20°$=altitude, $a'=31°$=altitude, φ=latitude, $\delta=15°$=declination, $\theta=15°$=sun's angular path for one hour, h=hour angle.

$$\therefore\ \cos h=\frac{\sin a-\sin\varphi\sin\delta}{\cos\varphi\cos\delta},\ \cos[h-\theta]=\frac{\sin a'-\sin\varphi\sin\delta}{\cos\varphi\cos\delta}.$$

Eliminating h, we get

$$[\sin a-\sin\varphi\sin\delta]\cos\theta+\{\cos^2\varphi\cos^2\delta-[\sin a-\sin\varphi\sin\delta]^2\}^{\frac{1}{2}}\sin\theta=\sin a'-\sin\varphi\sin\delta.$$

But $\theta=\delta$. $\therefore\ \sin^2\varphi\sin^2\delta[2+\cos\delta]-2\sin\varphi\sin\delta[1-\cos\delta][\sin a+\sin a']$

$$=\cos^2\delta\sin^2\delta+2\sin a\sin a'\cos\delta-\sin^2 a-\sin^2 a'.$$

$$\therefore\ .067055\sin^2\varphi-.015126\sin\varphi=.020553;\ \sin^2\varphi-.2256\sin\varphi=.3065.$$

$$\sin\varphi=.6778 \text{ or } -.4522;\ \varphi=42°\ 40'\ 30''.$$

115. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College Defiance, Ohio.

Determine geometrically where to stand so as to be able to throw a stone over a tree with the minimum velocity.

Solution by G. B. M. ZERR, A.M.. Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The velocity of projection is the same as a body would acquire in falling from the directrix of the parabolic path to the point of projection.

$\therefore$ The velocity will be a minimum when the directrix is the least distance above the top of the tree. This is the case when you stand at the base of the tree, then the directrix passes just above the tree.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

164. Proposed by JOSEPH V. COLLINS, Ph. D., Professor of Mathematics, State Normal School. Stevens Point, Wis.

Three women, the first with ten eggs, the second with thirty, and the third with fifty, went to market. They each got the same for their eggs, and all returned with the same money. What did they get?

ALGEBRA.

168. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

If n, $n+2$, $n+6$, $n+8$, $n+12$ are all primes, find the form of n.

169. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Solve $x^2+y+z=a\ldots.(1)$,
$x+y^2+z=b\ldots.(2)$,
$x+y+z^2=c\ldots.(3)$.

GEOMETRY.

191. Proposed by J. V. McADAMS, St. Louis, Mo.

Trisect any angle by means of the hypocycloid.

192. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Miss.

Of all triangles with a common base and inscribed in the same circle, the isosceles is the maximum and has the maximum perimeter. Prove geometrically.

CALCULUS.

158. Proposea by L. C. WALKER, A. M., Graduate Student, Leland Stanford University, Cal.

It is required to cut a hole a inches square, for a crank shaft, through the center of a grindstone b inches thick at the outer edge, c inches thick at the center, and d inches in diameter. How many cubic inches will have to be cut out?

159. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Solve $\dfrac{d^2u}{dx^2}=\dfrac{1}{m}\left(\dfrac{du}{dt}\right)$.

MECHANICS.

148. Proposed by G. H. HARVILL, A. M., Malakoff, Texas.

Show that a law of density for points in space may be assumed such that the joint mass of any two points which are *electrical images* of each other in respect to a given sphere may be constant, and that their centers of gravity should lie on the surface of the sphere.

149. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

From two points in the same horizontal line hangs a light inextensible string, on which are threaded two beads of equal mass. The beads start from rest in the position in which the terminal portions of the string are vertical and move symmetrically towards each other in the vertical plane. Find the path of each bead, and the tension of the string at any point in the path.

DIOPHANTINE ANALYSIS.

108. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

The determinant $\triangle$ of the special group-matrix for the Quaternion Group equals $\sigma_1\sigma_2\sigma_3\sigma_4(x_1-x_2)^4$, where $\sigma_1=x_1+x_2+2x_3+2x_5+2x_7$, $\sigma_2=x_1+x_2+2x_3-2x_5-2x_7$, $\sigma_3=x_1+x_2-2x_3+2x_5-2x_7$, $\sigma_4=x_1+x_2-2x_3-2x_5+2x_7$. Find the order of the linear group for the $GF[p^n]$, namely, the number of sets x_1, x_2, x_3, x_5, x_7, for which $\triangle$ is not equal to 0.

AVERAGE AND PROBABILITY.

133. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford University, Cal.

A circle of unknown radius is drawn with its center at the vertex of a given parabola, and has its greatest area when its circumference passes through the focus of the parabola. Required the average area common to the circle and parabola.

134. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

An ellipse, semi-axes a, b, is placed on a square, side c. Find the chance that center of ellipse is on the square.

MISCELLANEOUS.

130. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

From a cloud of angular elevation $\phi=45°$, a streak of lightning darted to the earth. The temperature of the atmosphere was $t=80°$, and the percentage of humidity $p=90$. After $m=3$ seconds, the report of the stroke at the earth was heard. How far away from the observer did the streak of lightning (1) start, and (2) strike the earth?

NOTES.

With this number of the MONTHLY begins a new arrangement for its future management which, we hope, will be as thoroughly appreciated by our readers as it is gratifying to us, and which, we are confident, will bring to it increased power and usefulness.

While in Chicago the first of September, we called on Dr. Dickson and urged him to join us in the editorship of the MONTHLY. Not seeing his way clear at the time, he withheld his answer until he could consider the matter. After some meditation, he decided affirmatively.

We feel that we are especially fortunate in securing the coöperation of so valuable a man as Dr. Dickson. He stands in the very front of the younger generation of mathematicians. Though not yet thirty years old, his contributions to mathematics in the way of original and important articles published in the various mathematical and scientific journals of the world is truly marvelous—

the most of his contributions being on the Theory of Groups. He is also the author of a *College Algebra* published by John Wiley & Sons; an *Introduction to Algebraic Equations* now being published by the same publishers, and a work on *Linear Groups*, published by B. G. Teubner, of Leipzig, Germany.

He is also on the editorial staff of the *Transactions* of the American Mathematical Society.

In 1896, Dr. Dickson took his doctor's degree at the University of Chicago; he then spent a year in study in Europe, one semester at Leipzig, and one semester at Paris. Returning to America, he became Instructor and later Assistant Professor of Mathematics in the University of California; then for one year Associate Professor of Mathematics in the University of Texas; and in 1900 he was called to the University of Chicago as Assistant Professor of Mathematics.

While Dr. Dickson has devoted the most of his time to research work and thereby materially aiding in enlarging the domain of mathematics, yet he is conscious of the importance of devoting time to the improvement of elementary texts, and thus placing the fundamentals of mathematics on a firm foundation. To this end he prepared his works on Algebra.

He thus brings to his work in the co-editorship of the MONTHLY a thorough and comprehensive knowledge of elementary as well as of higher mathematics. He will have full charge of the editing of all papers intended for publication in the MONTHLY. Therefore, we ask our contributors to send all articles for publication directly to him. His address is Box 142, Faculty Exchange, The University of Chicago.

All problems and solutions as well as all business communications pertaining to the MONTHLY should be sent, as heretofore, to B. F. Finkel, Springfield, Mo.

We hope that all our readers and contributors will coöperate with the editors in every way possible to make the MONTHLY a greater power for good than ever. B. F. F.

The hundredth anniversary of the birth of Abel, the eminent Norwegian mathematician, was celebrated during September at Christiania. Representative scientists from many countries were present. Among those upon whom honorary degrees were conferred were Simon Newcomb and J. Willard Gibbs. L. E. D.

Prof. J. M. Colaw, formerly associate editor of the MONTHLY, is coöperation with Prof. J. K. Ellwood, is now busily engaged in preparing for publication a text-book of Algebra. We are glad to note that the series of Arithmetics prepared by these gentlemen is meeting with wonderful success. B. F. F.

The recent address by John Purser, President of Section A of the British Association was an "Historical Sketch of the Irish School of Mathematics and Physics." It dealt at length on the life of Sir W. R. Hamilton. L. E. D.

Mr. J. W. Miller, Ph. D. (Columbia) has been appointed Instructor in Mathematics and Astronomy in Lehigh University. L. E. D.

BOOKS AND PERIODICALS.

Plane and Spherical Trigonometry. An elementary text-book. By Charles H. Ashton, A. M., Instructor in Mathematics in Harvard University, and Walter R. Marsh, A. B., Head Master Pingry School, Elizabeth, N. J. 8vo. Cloth, x+157 pages. New York: Charles Scribner's Sons.

Two distinguishing features of this text may be noted. The first is the systematic distinction (except in the first two chapters on the functions of acute angles and the solution of right triangles) between positive and negative directions of straight lines. The second is the proof of the addition theorem by the method of projection (as well as the usual proof). These features show that the text is thoroughly up to date. Among the minor points of pedagogical excellence is the *introduction* of the cosecant, secant, and cotangent as the reciprocals of the sine, cosine, and tangent, respectively. The book is well printed and the page attractive. L. E. D.

Elements of the Theory of the Newtonian Potential Functions. By B. O. Peirce, Ph. D., Hollis Professor of Mathematics and Natural Philosophy in Harvard University. Third Edition, Revised and Enlarged. 8vo. Cloth, xiii+490 pages. Price, $1.50. Boston and Chicago: Ginn & Co.

The chief difference between this and the first edition is in the way of additions. The first edition contained 178 pages, this contains 490 pages. In the present edition, a full discussion is given of electrostatics and electrokinematics. In this edition, 40 pages are devoted to a treatment of Electromagnetism, against none in the first edition. Current Induction is also more fully treated. Under each chapter is given a list of problems the solution of which will fix in the mind the principles under discussion. The work ends with a list of 387 problems some of which will tax the powers of the mathematician.

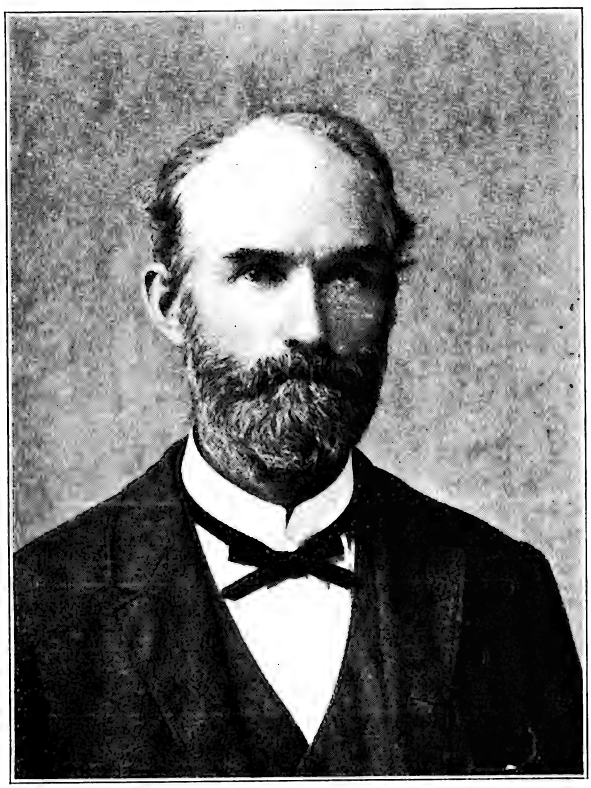
This scholarly book is a valuable contribution to mathematical Physics. It is very clear and easily understood by students who have a good working knowledge of mechanics and the calculus. B. F. F.

Elements of Physics. By Fernando Sanford, Professor in Leland Stanford Junior University. 8vo. Cloth, xxxi+426 pages. Price, $1.20. New York: Henry Holt & Co.

In this text-book, Mechanics is based on the energy concept from the beginning. In the treatment of properties of bodies, the gaseous state is treated before the liquid and the solid states. The author justifies this by the relative simplicity of the gaseous state of aggregation as compared with the liquid or solid states. Considerable stress is laid on the kinetic theory of gases. The greatest departure from established usage in elementary text-books is made in the subject of Optics. In this book Optics is treated geometrically from the beginning. This book contains only such experiments as are essential to the successful teaching of the subject. It is well written, but in the absence of a fairly well equipped laboratory, it is deficient in illustrations of important pieces of apparatus. B. F. F.

A Short Course in Plane and Spherical Trigonometry. By Edwin S. Crawley, Ph. D., Thomas A. Scott Professor of Mathematics in the University of Pennsylvania. 8vo. Cloth, 116+xxvii pages. Price, with tables, $1.00. Published by the author.

This is, in part, an abridgment of the author's "Elements of Plane and Spherical Trigonometry." It is most admirably adapted to the needs of the better class of High Schools and Normal Schools, and may be used to advantage in most colleges. B. F. F.



PHILETUS HARRY PHILBRICK.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. IX. NOVEMBER, 1902. No. 11.

A MATRIX DEFINED BY THE QUATERNION GROUP.

By DR. L. E. DICKSON.

1. The Quaternion Group was chosen by Weber* to illustrate his exposition of a portion of Frobenius's theory of group-matrices and group-determinants.† In his treatment of the illustrative example, Weber gives no clue to the reader how his results were obtained originally and makes the verification depend upon two laborious compositions of matrices of order eight. The example may, however, be treated very simply by a method which emphasizes certain remarkable properties enjoyed by group-matrices.

A second purpose of this paper is to furnish an instructive example of the theory of group-matrices as generalized for an arbitrary field (Körper, realm of rationality) by the writer.‡ The canonical forms are essentially different in the cases of a field having modulus 2 and a field not having modulus 2. The methods used are elementary and the paper is entirely independent of those cited. Incidentally, it illustrates a method of factoring important determinants.

2. The Quaternion Group may be defined by the multiplication-table§

*Weber, *Algebra*, edition of 1899, vol. 2, pp. 216-218; pp. 125-128.

†For an elementary, but complete, exposition of Frobenius's theory, see the writer's article in the *Annals of Mathematics*, October, 1902.

‡Dickson, *Transactions American Mathematical Society*, vol. 3 (1902), pp. 285-301; further developments in *Universi y of Chicago Decennial Publications*, vol. IX, pp. 35-51.

§The arrangement of the multipliers in the first column is such that the products equal to the identity element x_1 all appear in the main diagonal.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_1	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_2	x_2	x_1	x_4	x_3	x_6	x_5	x_8	x_7
x_4	x_4	x_3	x_1	x_2	x_7	x_8	x_6	x_5
x_3	x_3	x_4	x_2	x_1	x_8	x_7	x_5	x_6
x_6	x_6	x_5	x_8	x_7	x_1	x_2	x_3	x_4
x_5	x_5	x_6	x_7	x_8	x_2	x_1	x_4	x_3
x_8	x_8	x_7	x_5	x_6	x_4	x_3	x_1	x_2
x_7	x_7	x_8	x_6	x_5	x_3	x_4	x_2	x_1

The body of this table is a matrix M of order eight called the *group-matrix* of the Quaternion Group. Consider a transformation T on eight variables $\xi_1, \dots, \xi_8$ whose coefficients form the matrix M and are elements of a given field F.

3. Suppose first that the field F does not have modulus 2, so that division by 2 is possible in the field. An inspection of matrix M leads to certain linear functions which transformation T multiplies by constants:

$$\eta_1 = \xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 + \xi_7 + \xi_8, \quad \eta_3 = \xi_1 + \xi_2 - \xi_3 - \xi_4 + \xi_5 + \xi_6 - \xi_7 - \xi_8,$$
$$\eta_2 = \xi_1 + \xi_2 + \xi_3 + \xi_4 - \xi_5 - \xi_6 - \xi_7 - \xi_8, \quad \eta_4 = \xi_1 + \xi_2 - \xi_3 - \xi_4 - \xi_5 - \xi_6 + \xi_7 + \xi_8.$$

Indeed, T replaces them by the respective functions

$$(1) \qquad \eta_1' = \sigma_1 \eta_1, \quad \eta_2' = \sigma_2 \eta_2, \quad \eta_3' = \sigma_3 \eta_3, \quad \eta_4' = \sigma_4 \eta_4,$$

where the following abbreviations have been used:

$$\sigma_1 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8, \quad \sigma_3 = x_1 + x_2 - x_3 - x_4 + x_5 + x_6 - x_7 - x_8,$$
$$\sigma_2 = x_1 + x_2 + x_3 + x_4 - x_5 - x_6 - x_7 - x_8, \quad \sigma_4 = x_1 + x_2 - x_3 - x_4 - x_5 - x_6 + x_7 + x_8.$$

From the definitions of $\eta_1, \eta_2, \eta_3, \eta_4$, we get

$$\xi_1 + \xi_2 = \tfrac{1}{4}(\eta_1 + \eta_2 + \eta_3 + \eta_4), \quad \xi_5 + \xi_6 = \tfrac{1}{4}(\eta_1 - \eta_2 + \eta_3 - \eta_4),$$
$$\xi_3 + \xi_4 = \tfrac{1}{4}(\eta_1 + \eta_2 - \eta_3 - \eta_4), \quad \xi_7 + \xi_8 = \tfrac{1}{4}(\eta_1 - \eta_2 - \eta_3 + \eta_4).$$

Hence $\eta_1, \eta_2, \eta_3, \eta_4$ may be introduced as new independent variables in place of four of the original variables. Also, if we set

$$\varsigma_1 = \xi_1 - \xi_2, \quad \varsigma_2 = \xi_3 - \xi_4, \quad \varsigma_3 = \xi_5 - \xi_6, \quad \varsigma_4 = \xi_7 - \xi_8,$$

the functions $\eta_1, \eta_2, \eta_3, \eta_4, \varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4$ are evidently linearly independent functions of $\xi_1, \dots, \xi_8$ and may be introduced as new variables in place of the latter. Now T replaces $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4$ by the respective functions

$$(2)\qquad \begin{aligned} s_1' &= as_1+bs_2+cs_3+ds_4,\\ s_2' &= -bs_1+as_2+ds_3-cs_4,\\ s_3' &= -cs_1-ds_2+as_3+bs_4,\\ s_4' &= -ds_1+cs_2-bs_3+as_4, \end{aligned}$$

where

$$a=x_1-x_2,\ b=x_3-x_4,\ c=x_5-x_6,\ d=x_7-x_8.$$

It follows as before that $\sigma_1, \sigma_2, \sigma_3, \sigma_4, a, b, c, d$ are linearly independent functions of $x_1, x_2, \ldots, x_8$.

Let first the quantity $i\equiv\sqrt{-1}$ belong to the field F and set

$$\eta_5=s_1-is_2,\ \eta_6=s_3+is_4,\ \eta_7=-s_3+is_4,\ \eta_8=s_1+is_2,$$

so that $\eta_5, \eta_6, \eta_7, \eta_8$ are linearly independent functions of s_1, s_2, s_3, s_4. Then the transformation defined by (1) and (2) takes the canonical form

$$(3)\qquad \begin{cases} \eta_1'=\sigma_1\eta_1,\ \eta_3'=\sigma_3\eta_3,\ \eta_5'=\kappa\eta_5+\lambda\eta_6,\ \eta_7'=\kappa\eta_7+\lambda\eta_8,\\ \eta_2'=\sigma_2\eta_2,\ \eta_4'=\sigma_4\eta_4,\ \eta_6'=\mu\eta_5+\nu\eta_6, \eta_8'=\mu\eta_7+\nu\eta_8, \end{cases}$$

where

$$\kappa=a+ib,\ \lambda=c-id,\ \mu=-c-id,\ \nu=a-ib.$$

In particular, the determinant of transformation (3), and hence of matrix M, equals

$$\sigma_1\sigma_2\sigma_3\sigma_4(\kappa\nu-\lambda\mu)^2\equiv\sigma_1\sigma_2\sigma_3\sigma_4(a^2+b^2+c^2+d^2)^2.$$

As a corollary, we derive for the determinant of (2) the known formula

$$\begin{vmatrix} a & b & c & d\\ -b & a & d & -c\\ -c & -d & a & b\\ -d & c & -b & a \end{vmatrix}=(a^2+b^2+c^2+d^2)^2.$$

Suppose next that i does not belong to the field F. Then the transformation T of matrix M of coefficients belonging to F cannot be reduced to the canonical form (3) by means of a transformation of variables with coefficients in F. In fact, $a^2+b^2+c^2+d^2$ cannot be expressed in the form $\kappa\nu-\lambda\mu$, where $\kappa, \lambda, \mu, \nu$ are linear functions of a, b, c, d with coefficients in F (not containing i). As the canonical form of T for the field F, we therefore take the transformation defined by (1) and (2). For the enlarged field $F(i)$, obtained by adjoining i to F, we may take as the canonical form either (3) or, preferably, the transformation

$$(3')\qquad \begin{cases} \eta_1'=\sigma_1\eta_1,\ \eta_3'=\sigma_3\eta_3,\ \eta_5'=\kappa\eta_5+\lambda\eta_6,\ \ \eta_7'=\kappa\eta_7+\lambda\eta_8,\\ \eta_2'=\sigma_2\eta_2,\ \eta_4'=\sigma_4\eta_4,\ \eta_6'=-\bar\lambda\eta_5+\bar\kappa\eta_6,\ \eta_8'=-\bar\lambda\eta_7+\bar\kappa\eta_8, \end{cases}$$

where $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are arbitrary quantities in F, while κ, λ are arbitrary quantities in $F(i)$, $\bar{\kappa}$ denoting the conjugate of κ.

4. We may now readily determine the order of the group* of all transformations of matrix M of coefficients in the Galois Field of order p^n, $p>2$.

Let first i belong to the $GF[p^n]$, so that p^n-1 is a multiple of 4, the period of i. We have then to determine the number of the transformations (3) of determinant not zero. Here $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \kappa, \lambda, \mu, \nu$ are any elements of the $GF[p^n]$ such that

$$(4) \qquad \sigma_1\sigma_2\sigma_3\sigma_4(\kappa\nu-\lambda\mu)^2\neq 0,$$

so that the order of the group is $(p^n-1)^4.(p^{2n}-1)(p^{2n}-p^n)$.

Let next i be not in the $GF[p^n]$, so that it serves to extend the $GF[p^n]$ to the $GF[p^{2n}]$. We have then to determine the number of transformations (3') of the determinant not zero. Here $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are any elements $\neq 0$ of the $GF[p^n]$, while κ and λ are any elements of the $GF[p^n]$ such that

$$\triangle \equiv \kappa\bar{\kappa}+\lambda\bar{\lambda}\equiv \kappa^{p^n+1}+\lambda^{p^n+1}\neq 0.$$

But $\triangle=0$ gives $\kappa=\tau\lambda$, where $\tau^{p^n+1}=-1$. Hence there are

$$1+(p^{2n}-1)(p^n+1)$$

sets κ, λ making $\triangle=0$. Subtracting this sum from p^{4n}, the total number of sets κ, λ in the $GF[p^{2n}]$, we obtain $(p^{2n}-1)(p^{2n}-p^n)$ as the number of sets κ, λ making $\triangle\neq 0$. Hence the order is again†

$$(p^n-1)^4.(p^{2n}-1)(p^{2n}-p^n).$$

5. Suppose finally that the field F has modulus 2 The four functions in §3 which T multiplies by constants are now identical. We seek eight linear functions $\eta_1, \ldots, \eta_8$ which are linearly independent functious modulo 2 of

*In view of the canonical form (3) or (3'), the totality of the transformations T evidently possesses the group property.

†Another method, not assuming the independence of $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \kappa, \lambda, \mu, \nu$, consists in finding the number of sets $x_1, \ldots, x_8$ for which

$\sigma_1\neq 0,\ \sigma_2\neq 0,\ \sigma_3\neq 0,\ \sigma_4\neq 0,\ \kappa\nu-\lambda\mu=0$	$p^n(p^{2n}+p^n-1)(p^n-1)^4$
$\sigma_1\neq 0,\ \sigma_2\neq 0,\ \sigma_3\neq 0,\ \sigma_4=0$	$p^{4n}(p^n-1)^3$
$\sigma_1\neq 0,\ \sigma_2\neq 0,\ \sigma_3=0$	$p^{5n}(p^n-1)^2$
$\sigma_1\neq 0,\ \sigma_2=0$	$p^{6n}(p^n-1)$
$\sigma_1=0$	p^{7n}

Subtracting the sum of the numbers in the last column from p^{8n}, the total number of sets $x_1, \ldots, x_8$, we obtain $p^n(p^n+1)(p^n-1)^6$ as the number of sets $x_1, \ldots, x_8$ for which (4) holds, and hence equals the order of the group.

$\xi_1, \ldots, \xi_8$ such that T replaces each η_i by a linear function of $\eta_1, \ldots, \eta_i$. The desired functions are

$$\eta_1=\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6+\xi_7+\xi_8, \quad \eta_2=\xi_1+\xi_2+\xi_3+\xi_4,$$
$$\eta_3=\xi_1+\xi_2+\xi_7+\xi_8, \qquad \eta_4=\xi_1+\xi_2, \quad \eta_5=\xi_1+\xi_3+\xi_5+\xi_7,$$
$$\eta_6=\xi_1+\xi_3, \qquad \eta_7=\xi_1+\xi_5, \quad \eta_8=\xi_1.$$

To prove them linearly independent modulo 2, we note that linear combinations of $\eta_1, \eta_2, \eta_3, \eta_4$ give $\xi_1+\xi_2$, $\xi_3+\xi_4$, $\xi_5+\xi_6$, $\xi_7+\xi_8$; while linear combinations of $\eta_5, \eta_6, \eta_7, \eta_8$ give $\xi_1, \xi_3, \xi_5, \xi_7$ We find that T gives rise to the following transformation:

$$\begin{aligned}
\eta_1'&=\rho_1\eta_1,\\
\eta_2'&=\rho_2\eta_1+\rho_1\eta_2,\\
\eta_3'&=\rho_3\eta_1+\rho_1\eta_3,\\
\eta_4'&=\rho_4\eta_1+\rho_3\eta_2+\rho_2\eta_3+\rho_1\eta_4,\\
\eta_5'&=\rho_5\eta_1+(\rho_2+\rho_3)\eta_2+\rho_3\eta_3+\rho_1\eta_5,\\
\eta_6'&=\rho_6\eta_1+(\rho_3+\rho_4+\rho_5)\eta_2+\rho_4\eta_3+\rho_3\eta_4+\rho_1\eta_6,\\
\eta_7'&=\rho_7\eta_1+(\rho_2+\rho_3+\rho_5)\eta_2+(\rho_3+\rho_4+\rho_5)\eta_3+\rho_2\eta_4+(\rho_2+\rho_3)\eta_5+\rho_1\eta_7,\\
\eta_8'&=\rho_8\eta_1+(\rho_3+\rho_4+\rho_5+\rho_6+\rho_7)\eta_2+\rho_6\eta_3+(\rho_3+\rho_4+\rho_5)\eta_4+(\rho_2+\rho_4)\eta_5+(\rho_2\\
&\qquad+\rho_3)\eta_6+\rho_2\eta_7+\rho_1\eta_8,
\end{aligned}$$

the coefficients having the following values:

$$\rho_1=x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8, \quad \rho_2=x_5+x_6+x_7+x_8,$$
$$\rho_3=x_3+x_4+x_5+x_6, \qquad \rho_4=x_5+x_6, \quad \rho_5=x_2+x_3+x_6+x_8,$$
$$\rho_6=x_6+x_8, \qquad \rho_7=x_2+x_6, \quad \rho_8=x_6.$$

The determinant of the transformation is evidently ρ_1^8. The only condition on $\rho_1, \ldots, \rho_8$ is, therefore, that $\rho_1 \neq 0$. Hence the order of the group in the $GF[2^n]$ is $2^{7n}(2^n-1)$.

Since the expressions for $\eta_1', \eta_2', \eta_3', \eta_4'$ remain unaltered by the interchange of the variables η_i with the parameters ρ_i, we have a four-parameter *commutative subgroup* on the variables $\eta_1, \eta_2, \eta_3, \eta_4$.

A five-parameter commutative subgroup is obtained by setting $x_3=x_4$, $x_5=x_6$, $x_7=x_8$, so that $\rho_2=\rho_3=\rho_4\equiv 0 \pmod 2$. Also

$$\rho_1=x_1+x_2; \quad \rho_5=x_2+x_3+x_6+x_8, \quad \rho_6=x_6+x_8, \quad \rho_7=x_2+x_6, \quad \rho_8=x_6$$

are linearly independent functions of x_1, x_2, x_3, x_6, x_8 modulo 2. The above transformation now becomes

$$\begin{aligned}
&\eta_1'=\rho_1\eta_1, \quad \eta_5'=\rho_1\eta_5+\rho_5\eta_1,\\
&\eta_2'=\rho_1\eta_2, \quad \eta_6'=\rho_1\eta_6+\rho_6\eta_1 \qquad +\rho_5\eta_2,\\
&\eta_3'=\rho_1\eta_3, \quad \eta_7'=\rho_1\eta_7+\rho_7\eta_1 \qquad +\rho_5\eta_2+\rho_5\eta_3,\\
&\eta_4'=\rho_1\eta_4, \quad \eta_8'=\rho_1\eta_8+\rho_8\eta_1+(\rho_5+\rho_6+\rho_7)\eta_2+\rho_6\eta_3+\rho_5\eta_4.
\end{aligned}$$

Denoting this transformation by T_ρ, we verify that $T_\rho\, T_{\rho'}=T_{\rho''}$, where

$$\begin{aligned}
&\rho_1''=\rho_1\rho_1', \ \rho_5''=\rho_5\rho_1'+\rho_1\rho_5', \ \rho_6''=\rho_6\rho_1'+\rho_1\rho_6',\\
&\rho_7''=\rho_7\rho_1'+\rho_1\rho_7', \ \rho_8''=\rho_8\rho_1'+\rho_1\rho_8'.
\end{aligned}$$

Hence the transformations T_ρ form a group and the group is commutative. As this result is in accord with the general theory, we have a complete check upon the above work.

The University of Chicago, October 10, 1902.

THE LENGTH OF THE CIRCLE.

By DR. GEORGE BRUCE HALSTED.

The simple theorem of Euclid (Eu. I. 20) that any two sides of a triangle are together greater than the third, appears in popular geometries as a result, a deduction from the definition: the straight line is the shortest distance between two points. But such a definition is, as Hilbert says, *senseless* if one has not defined the concept length.

Without presupposing the idea of the length of the curve, it cannot use anything but the straight, and so is merely a *petitio principii.*

In fact, as Hilbert points out, the essential content of the statement, the straight line is the shortest distance between two points, reduces merely to this theorem of Euclid.

The same fallacy lurks in the theorem, "An arc of a circle is less than any line which envelops it and has the same extremities," recently 'borrowed' by Wentworth (1899, p. 219, Book V, Proposition VII, §451) from Chauvenet (1869, 1881, p. 155, Book V, Proposition XII, §32). He says "Of all the lines, etc., there must be at least one shortest line."

But what is the length of a curve? Again we have a begging of the question. Upon these two holes, empty places, or chasms, he then attempts to rest a demonstration that the length of a circle is the limit of the lengths of the perimeters of inscribed polygons.

The sooner this procedure is banished from elementary geometry the better for the world.

Fortunately there is here no need to touch that very difficult general question, what is the length of a curve in general.

We have only for consideration the simplest of curves, the circle.

For it then only need we show the simple theorem (Halsted's *Mensuration*, p. 16): When their sides tend indefinitely towards zero, the perimeter of the polygon inscribed increases, circumscribed decreases, toward the same limit.

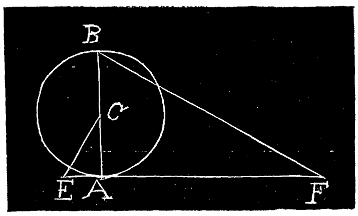
After that, we may deliberately and consciously declare by *definition* that this limit shall be what we will mean when we use the word *length* in connection with *circle*.

Here is no attempt to *prove* that the length of the circle is the limit of the lengths of perimeters of polygons. Any such attempt presupposes that we already know in some other way, or have in some other mathematical way defined what we are then already to mean by the length of the circle before we try to prove it equivalent to the limit for perimeters. What text book does this?

Can it be done? Let me try. (*Sect* is English for the German *strecke*).

Definition of length of an arc. We assume that with every arc is connected a sect such that if an arc be cut into two arcs, this sect is the sum of their sects; moreover this sect is not less than the chord of the arc, nor, if the arc be minor, is it greater than the sum of the sects on the tangents from the extremities of the arc to their intersection.

This sect we call *the length of the arc.*



Kochansky (1685) gave the following simple construction for the approximate length of the semicircle.

At the end-point A of the diameter BA, draw the tangent to the circle with center C. Take $ACE=\frac{1}{3}$ right angle. On the tangent, take $EF=3AC$. Then BF is with great exactitude the length of the semicircle.

In fact, $BF=r[13\frac{1}{2}-2(3)^{\frac{1}{2}}]^{\frac{1}{2}}=r3.1415$.

University of Texas, November, 1902.

AN ELEMENTARY ACCOUNT OF THE PICARD-VESSIOT THEORY.

By DR. SAUL EPSTEEN.

A theory of linear differential equations has been built up within the last twenty years which resembles the Galois theory of algebraic equations very closely.

In the Galois theory of equations we begin with an algebraic equation

$$(1) \qquad x^n+a_1x^{n-1}+\dots+a_n=0$$

and study the $n!$ permutations of the n roots $x_1, \dots, x_n$.

In the corresponding theory of differential equations we begin with the linear homogeneous differential equation

$$\text{(I)} \qquad \frac{d^n y}{dx^n}+p_1\frac{d^{n-1}y}{dx^{n-1}}+\ldots.+p_n y=0$$

and study the transformations of the continuous group

$$y'_i=\sum_{k=1}^{n} a_{ik}y_k \quad (i=1, \ldots., n),$$

where $y_1, \ldots., y_n$ form a fundamental system of integrals.

Many theorems analogous to well known theorems of algebra can be demonstrated in this new theory. For example, the theorem of Lagrange for indeterminate roots: "If a rational function $\phi(x_1, \ldots., x_n)$ remains unaltered by all the substitutions which leave another rational function $\psi(x_1, \ldots., x_n)$ unaltered, then ϕ can be expressed rationally in terms of ψ; $\phi=$Rat. $(\psi, a_1, \ldots., a_n)$ has the following analog: If a rational function $\phi(y_1, \ldots., y_n)$ of the integrals remains invariant under all the transformations which constitute the group of another rational function $\psi(y_1, \ldots., y_n)$ then ϕ is rationally expressible in terms of ψ; $\phi=$Rat. $(\psi, p_1, \ldots., p'_1, \ldots., x)$."

As is well known, the group of an algebraic equation is defined by the two fundamental properties:

(a) Every rational function of the roots, which remains unaltered by all the substitutions of the group, is rationally known.

(b) Every rational function of the roots, which is rationally known, remains unaltered by all the substitutions of the group.

The parallel defining properties of the group of a differential equation are:

(A) Every rational function of the integrals (and their derivatives), which remains unaltered by all the transformations of the group, is rationally known.

(B) Every rational function of the integrals (and their derivatives), which is rationally known, remains unaltered by all the transformations of the group.

This brings us now to the question of formal and numerical invariance which is very much alike in both theories. The question will be illustrated by examples.

Let the algebraic equation be $x^4+1=0$ with the roots

$$x_1=\varepsilon,\ x_2=i\epsilon,\ x_3=-\varepsilon,\ x_4=-i\varepsilon \quad \left(\epsilon=\frac{1+i}{\sqrt{2}}\right).$$

The group of this equation is*

$$G=[\mathrm{I};\ (x_1x_2)(x_3x_4);\ (x_1x_3)(x_2x_4);\ (x_1x_4)(x_2x_3)].$$

*Bolza, *Bulletin American Mathematical Society*, First Series, Vol. 2, p. 99.

For the domain of rational numbers, the function $x_1 x_2$ has the rational value -1. The substitutions of G which leave $x_1 x_2$ formally invariant are $[\mathrm{I}; (x_1 x_2)(x_3 x_4)]$, which form only a subgroup of the group of the equation. But all the substitutions of G leave $x_1 x_2$ numerically invariant, since $x_1 x_2 = x_3 x_4 = -1$.*

Likewise in differential equations. The integral being regarded as functions of x, namely, $y_i = y_i(x)$ $(i=1, 2, \ldots, n)$, we consider, not the continuous group which leaves some function such as $\varphi(y_1, \ldots, y_n) = r(x)$ invariant as a function of the y's, but we consider a group which may change $\varphi(y_1, \ldots, y_n)$ to another function $\psi(y_1, \ldots, y_n)$ provided that *the numerical value is unaltered:* $\varphi = \psi = r(x)$. The value of x is not specified.

As an illustration, consider an irreducible linear homogeneous differential equation of the third order,

$$\frac{d^3 y}{dx^3} + p_1 \frac{d^2 y}{dx^2} + p_2 \frac{dy}{dx} + p_3 y = 0,$$

between a fundamental set of whose integrals there exists the *simple* rational relation

$$\varphi(y_1, y_2, y_3) \equiv y_2^2 - y_1 y_3 = 0.$$

If there were several such relations (as may well happen for equations of higher order) we must consider the group which leaves them all simultaneously invariant numerically.

The 3-parameter group

$$\begin{aligned} & y_1' = a^2 y_1 + 2ab y_2 + b^2 y_3 \\ (2) \qquad & y_2' = ac y_1 + (ad+bc) y_2 + bd y_3 \\ & y_3' = c^2 y_1 + 2cd y_2 + d^3 y_3 \end{aligned}$$

with $ad - bc = 1$, leaves φ formally invariant since

$$y_2'^2 - y_1' y_3' \equiv y_2^2 - y_1 y_3.$$

But it is not the group of the equation. The group of the equation (in the domain of the coefficients) is the 4-parameter group (2) where $ad - bc =$ any constant different from zero. This latter group does not leave φ formally invariant; for now

$$y_2'^2 - y_1' y_3' = (ad - bc)^3 (y_2^2 - y_1 y_3).$$

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*From this it must not be concluded that all the substitutions which leave $x_1 x_2$ numerically unchanged: $G_8 = [\mathrm{I}; (x_1 x_2); (x_3 x_4); (x_1 x_2)(x_3 x_4); (x_1 x_3)(x_2 x_4); (x_1 x_4)(x_2 x_3); (x_1 x_3 x_2 x_4); (x_1 x_4 x_2 x_3)]$ form the group of the equation. The reason for this lies in (*b*); for we must consider not merely G_8 which leaves the *single* function $x_1 x_4 = -1$ numerically invariant, but we must consider a group (which turns out to be G) which leaves *every* function which has a rational value, unaltered. Thus $x_1 x_2^2 + x_3 x_4^2 = 0$ has a rational value. Now the transposition $(x_1 x_2)$ of G_8 changes this to $x_2 x_1^2 + x_3 x_4^2 = \varepsilon(i-1)$, therefore G_8 is not the group of the equation. Notice that the group of the equation, G, will actually leave $x_1 x_2^2 + x_3 x_4^2 = 0$ numerically unaltered.

The numerical value zero (for $\varphi=0$) is however unaltered by the group of the equation. (If φ were equal to $r(x)$ instead of to zero, the group of the equation would be (2) with $ad-bc=1$; for $ad-bc\neq 1$, the numerical value would change from $r(x)$ to $(ad-bc)^3 r(x)$.)

In the above example we assumed that the function $\varphi=y_2^2-y_1 y_3$ was the invariant of the group. If the differential equation was of a higher order than the third the relation would be more complicated. Indeed, we might have a number of such relations involving not only the integrals but also their derivatives, say $\varphi_i(y_1 \ldots y_n, y_1' \ldots y_n, y_1'' \ldots)$, $(i=1 \ldots p)$. The generalization can be carried still further. Notice that we assumed above that there exists algebraic relations between the integrals and their derivatives. Now this assumption is unnecessary, it may well happen that there are no such relations whatever, yet we continue still to speak of the group of the equation. An exposition of this question would be beyond the range set for this paper.

The group in question is technically called the *group of rationality* of the equation.

The literature of this subject is now quite extensive, but anyone familiar with the elements of the Galois theory will find it very easy to read the brief summary in Picard's *Traité de Analyse* III, last chapter, and to supplement it with Klein's *Höhere Geometrie* II, pp. 298-9.

The University of Chicago, October, 1902.

TWO SIMPLE CONSTRUCTIONS FOR FINDING THE FOCI OF AN HYPERBOLA, GIVEN THE ASYMPTOTES AND A POINT ON, OR A TANGENT TO, THE CURVE.

By ARCHIBALD HENDERSON, Ph. D., Associate Professor of Mathematics, University of North Carolina.

The construction for a special position of the point is given first, as it is a linear construction.

Given the asymptotes of a hyperbola and the vertex A of the curve, to construct the foci.

The major axis is fixed, bisecting the angle between the asymptotes. Lay off, along an asymptote, from the center C of the hyperbola a distance $CD=CA=a$; draw a perpendicular to the asymptote at D. This meets the major axis (produced) in a focus F_1 (F_2).

For, the perpendicular F_1D from the focus $(ae, 0)$ upon the asymptote, whose equation is $y=\dfrac{b}{a}x$, is equal to b (by elementary principles). But $CF_1=\sqrt{(a^2+b^2)}$ and therefore $CD=a$.

The following style of argument (communicated to me in a letter a few

and the points F_1, E, F_2, D are concyclic. The perpendicular bisectors of F_1F_2 and DE therefore intersect at C, the required center.

It may be observed that the second construction is a direct generalization of the one (when the asymptotes and vertex are given) found in all elementary text-books on Analytic Geometry. Would it not be well for the latter to contain the above simple constructions?

The University of Chicago, October, 1902.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

161. Proposed by F. M. SHIELDS, Coopwood, Miss.

If 1 man, 1 boy, and 1 girl catch 1 trout, 1 perch, and one minnow in 5 minutes, and 1 man, 2 boys, and 3 girls catch 1 trout, 2 perch, and 3 minnows in 6 minutes, how many minutes will be required for 2 men, 3 boys, and 4 girls to catch 5 trout, 11 perch, and 17 minnows?

Remarks by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

This problem is indeterminate. Let A_1, A_2, and A_3 be the activities of a man, boy, and girl, respectively; and E_1, E_2, and E_3 the work required to catch a trout, a perch, and a minnow, respectively. Then

$$5(A_1+A_2+A_3)=E_1+E_2+E_3\dots\dots(1), \text{ and}$$
$$6(A_1+2A_2+3A_3)=E_1+2E_2+3E_3\dots\dots(2).$$

From these two conditions it is required to find t so that the following condition is satisfied, viz:

$$t(2A_1+3A_2+4A_3)=5E_1+11E_2+17E_3.$$

This is clearly impossible in general. By assuming certain other conditions as, for example, the ratios of the activities, and the ratios of the work, a solution may be effected.

162. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A trolly road is built between two towns, and it is found that the gross annual receipts amount to 20% of the original cost; the annual cost of repairs 2% of the original cost; and the working expense is \$3000 in addition to 20% of the net receipts. After a year a second road is built at the same cost as the first and it is found that the gross re-

ceipts and working expenses per year are doubled, while the cost of repairs for the new road is 1% of cost. If the receipts for both roads is \$72,500, find the cost of each road, and the net receipts the first year.

Solution by the PROPOSER.

Let 100%=cost of each road, 20%=gross receipts first year, 2%=repairs first year.

Working expenses=\$3000+20% of (20%−2%−working expenses).

$\therefore$ $\frac{6}{5}$ working expenses=\$3000+3.6%.

$\therefore$ working expenses=\$2500+3%.

The second year, gross receipts=40%, repairs=3%, working expenses= \$5000+6%.

$\therefore$ \$72,500=40%−3%−6%−\$5000.

$\therefore$ 31%=\$77,500, 100%=\$250,000, cost of each road.

Net receipts first year=20%−2%—\$2500−3%=15%—\$2500=\$35,000.

Also solved by *W. A. CLEMENSER*, Heidelberg University, Tiffin, O.; and *C. A. SHORT*, Assistant Professor of Mathematics, Delaware College, Delaware.

ALGEBRA.

147. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Prove that $x=a^x$ has never more than two real roots, and find the condition for no real roots.

Solution by the PROPOSER.

$x=a^x$. $\therefore$ $a=x^{1/x}$. $a_1=\frac{x^{1/x}}{x^2}(1-\log x)$ where $a_1=\frac{da}{dx}$ and $a_1=0$ for $x=0$, e, ∞. For $x>1$, a is >1; for $0<x<1$, $0<a<1$, for $x=1$, $a_1=1$.

$\therefore$ The curve represented by the equation touches axis of x at $x=0$, and its ordinates increase from 0 to 1 with its abscissæ; at $x=1$ it is touching the bisector of the axes. From 1 to e for x, ordinates increase from 1 to $e^{1/e}$, and as x increases from e to ∞, the ordinates decrease from $e^{1/e}$ to 1.

$\therefore$ For $0<a<1$, a real root between 0 and 1,

$1<a<e^{1/e}$, a real root between 1 and $e^{1/e}$, or another between e and ∞.

$a=e^{1/e}$, the root is e.

If $a>1$ and $re^{\theta i}$ an imaginary root, then if $ra.\theta=u$, $r\sin\theta=v$ we have $re^{\theta i}=e^{\log a re^{\theta i}}=e^{\log(a+\theta i)}$ and the equation becomes, on equating real and imaginary parts, $\log r=u\log a$; $\theta=v\log a$. The intersection of these curves gives the imaginary roots required, but we must reject roots between the lines $v\log a=(2k+1)\pi$ and $v\log a=(2k+2)\pi$. If $a<1=1/b$ the curves are $r=b^{-u}$, $\theta=-v\log b$, and these curves are the images of the two previous, and the branches giving the roots of the equation proposed are the images of the branches which we previously rejected, and vice versa.

For a full discussion of this equation see Nouville's *Annales*, 1896, p. 548; 1897, p. 54.

158. Proposed by R. D. BOHANNON, Ph. D., Professor of Mathematics, Ohio State University, Columbus, O.

If $\frac{x}{a+\alpha}+\frac{y}{b+\alpha}+\frac{z}{c+\alpha}=\frac{x}{a+\beta}+\frac{y}{b+\beta}+\frac{z}{c+\beta}=\frac{x}{a+\gamma}+\frac{y}{b+\gamma}+\frac{z}{c+\gamma}=1$, show,

$$\frac{x}{(a+\beta)^2}+\frac{y}{(b+\beta)^2}+\frac{z}{(c+\beta)^2}=\frac{(\gamma-\beta)(\alpha-\beta)}{(a+\beta)(b+\beta)(c+\beta)}.$$

Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Consider the equation,

$$\frac{x}{a+\varphi}+\frac{y}{b+\varphi}+\frac{z}{c+\varphi}=1-\frac{(\varphi-\alpha)(\varphi-\beta)(\varphi-\gamma)}{(a+\varphi)(b+\varphi)(c+\varphi)}.$$

Multiply through by $a+\varphi$ and then put $a+\varphi=0$.

$$\therefore x=\frac{(a+\alpha)(a+\beta)(a+\gamma)}{(a-b)(a-c)},\quad y=\frac{(b+\alpha)(b+\beta)(b+\gamma)}{(b-c)(b-a)},\quad z=\frac{(c+\alpha)(c+\beta)(c+\gamma)}{(c-a)(c-b)}$$

$$\therefore \frac{x}{(a+\beta)^2}+\frac{y}{(b+\beta)^2}+\frac{z}{(c+\beta)^2}=\frac{(a+\alpha)(a+\gamma)}{(a+\beta)(a-b)(a-c)}$$

$$+\frac{(b+\alpha)(b+\gamma)}{(b+\beta)(b-c)(b-a)}+\frac{(c+\alpha)(c+\gamma)}{(c+\beta)(c-a)(c-b)}=\frac{(\gamma-\beta)(\alpha-\beta)}{(a+\beta)(b+\beta)(c+\beta)}.$$

Also $\frac{x}{(a+\alpha)^2}+\frac{y}{(b+\alpha)^2}+\frac{z}{(c+\alpha)^2}=\frac{(\gamma-\alpha)(\beta-\alpha)}{(a+\alpha)(b+\alpha)(c+\alpha)}$.

$$\frac{x}{(a+\gamma)^2}+\frac{y}{(b+\gamma)^2}+\frac{z}{(c+\gamma)^2}=\frac{(\alpha-\gamma)(\beta-\gamma)}{(a+\gamma)(b+\gamma)(c+\gamma)}.$$

Also solved by *LON C. WALKER.*

159. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

If $x-1=3m$, $x^2-1=4n$, $x^3-1=5p$, where m, n, p are integers, find a general expression for x.

I. Solution by L. E. DICKSON, Ph. D., The University of Chicago, Chicago, Ill.

The required expression for x is of the form $3m+1$, where m is subject to the conditions that $(3m+1)^2-1$ shall be divisible by 4, and that $(3m+1)^3-1$ shall be divisible by 5. The first condition is satisfied if, and only if, m is even. The second condition requires that either m or else $3m^2+3m+1$ shall be divisible by 5. To prove that the last alternative is impossible, we note that

$$2(3m^2+3m+1)\equiv(m+3)^2-2 \pmod 5),$$

and hence is not$\equiv 0 \pmod 5$, 2 being a non-quadratic residue of 5. The necessary and sufficient conditions are, therefore, that m shall be divisible by 10, so that x shall have the form $x=30t+1$.

II. Solution by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

From the three equations we readily obtain

$$x-1=3m \qquad\qquad =3m,$$
$$x^2-1=3m(2+3m) \qquad =4n,$$
$$x^3-1=9m[1+3m(1+m)]=5p;$$
$$\text{whence } n=\tfrac{3}{4}m(2+3m) \text{ and } p=\tfrac{9}{5}m[1+3m(1+m)],$$

where m must be so taken as to make n and p integers. To make n integral m must be an *even* number; to make p integral m must be a multiple of 5; hence $m=10k$ where k may be any positive integer. Hence

$$m=10k,$$
$$n=15k(1+15k),$$
$$p=18k[1+30k(1+10k)].$$
$$\text{Also } x-1=30k,$$
$$x^2-1=60k(1+15k),$$
$$x^3-1=90k[1+30k(1+10k)];$$
$$\text{and } x=1,\ 31,\ 61,\ 91, \text{ etc.}$$

Solved similarly by *G. B. M. ZERR*, *LON C. WALKER*, and *HON. J. H. DRUMMOND*.

160. Proposed by J. SCHEFFER, A. M., Hagerstown, Mo.

Represent the square root of $10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15}$ as the sum of three square roots.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; LON C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University; and the late HON. JOSIAH H. DRUMMOND.

Let $\sqrt{[10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15}]}=\sqrt{x}+\sqrt{y}+\sqrt{z}$.

$\therefore\ 10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15}=x+y+z+2\sqrt{xy}+2\sqrt{xz}+2\sqrt{yz}$.

$\therefore\ 2\sqrt{xy}=2\sqrt{6}$, $\therefore\ \sqrt{xy}=\sqrt{6}$. Similarly, $\sqrt{xz}=\sqrt{10}$, $\sqrt{yz}=\sqrt{15}$.

$\therefore\ \sqrt{(xyz)}=\sqrt{30}$. $\therefore\ \sqrt{x}=\sqrt{2}$, $\sqrt{y}=\sqrt{3}$, $\sqrt{z}=\sqrt{5}$.

$\therefore\ \sqrt{[10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15}]}=\sqrt{2}+\sqrt{3}+\sqrt{5}=$the sum of the three square roots.

GEOMETRY.

187. Proposed by G. TUCKER, M. A.

AD, BE, CF are the altitudes of the triangle ABC; k_1, k_1'; k_2, k_2'; k_3, k_3' are the S points of the triangles EAB, FCA; FBC, DCB; DCA, EBC, respectively; prove that $k_3'k_1=k_4'k_2=k_2'k_3=R\sin A\sin B\sin C$. ρ_1, ρ_1'; ρ_2, ρ_2'; $\rho_3\rho_3'$ are the Brocard radii of the above triangles, prove that (1) $\rho_1\rho_2\rho_3=\rho_1'\rho_2'\rho_3'$; (2) $(\rho_2'^2-\rho_3^2)/a^2+(\rho_3'^2-\rho_1^2)/b^2+(\rho_1'^2-\rho_2^2)/c^2=\frac{3}{64}$; (3) the sets of 4 Brocard-points for the above pairs of triangles are concyclic (on three circles); (4) the tangents from any one of the right angles of the above triangles to the Brocard circle of the tangent is a mean proportional between the tangents to the same circle from the remaining (two) angles.

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The S point of each triangle is found by drawing from its right angle a line perpendicular to its opposite side, and taking its mid-point. Let k_1, k_2, and k_3 be the mid-points of HF, FK, and DI, respectively, and k_1', k_2', k_3', the mid-points of GE, LD, and JE, respectively. From the inscribed quadrilateral $BGEJ$ we get $GB.JE+BJ.GE=GJ.BE$, $JE\sin A+GE\sin C=GJ$.

$\therefore$ $\sin A\sin C(a\cos C+c\cos A)=GJ=b\sin A\sin C$.

Similarly, $KH=c\sin A\sin B$, $LI=a\sin B\sin C$.

$\therefore$ $GJ=KH=LI=2R\sin A\sin B\sin C$.

$\therefore$ $k_3'k_1=\frac12 GJ=k_1'k_2=\frac12 KH=k_2'k_3=\frac12 LI=R\sin A\sin B\sin C$.

(1) Let ω_1, ω_1', ω_2, ω_2', ω_3, ω_3' be the Brocard angles; then $\omega_1=\omega_1'$, $\omega_2=\omega_2'$, $\omega_3=\omega_3'$, for $\cot\omega_1=\cot\omega_1'=\cot A+\tan A$.

$\therefore$ $b\rho_1=c\rho_1'=\frac14 bc(1-3\cos^2 A\sin^2 A)^{\frac12}$, $c\rho_2=a\rho_2'=\frac14 ac(1-3\cos^2 B\sin^2 B)^{\frac12}$, $a\rho_3=b\rho_3'=\frac14 ab(1-3\cos^2 C\sin^2 C)^{\frac12}$.

$\therefore$ $\rho_1\rho_2\rho_3=\rho_1'\rho_2'\rho_3'$, $(\rho_2'^2-\rho_3^2)a^2+(\rho_3'^2-\rho_1^2)b^2+(\rho_1'^2-\rho_2^2)c^2=0$.

(2) $$(\rho_2'^2-\rho_3^2)/a^2+(\rho_3'^2-\rho_1^2)/b^2+(\rho_1'^2-\rho_2^2)/c^2=\frac{c^2-b^2}{16a^2}+\frac{a^2-c^2}{16b^2}$$
$$+\frac{b^2-a^2}{16c^2}-\frac{3\triangle^4}{a^2b^2b^2}\left[\frac{c^2-b^2}{a^4}+\frac{a^2-c^2}{b^4}+\frac{b^2-a^2}{c^4}\right].$$

(3) Let M, M_1 be the Brocard-points of EAB; M_2, M_3 those of FCA. Since $\omega_1=\omega_1'$, AM_3M and AM_1M_2 are three each on a straight line.

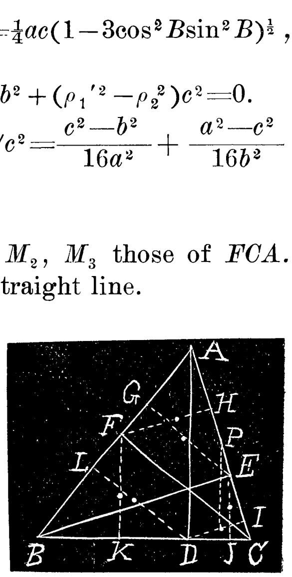
But $AM=c\sin\omega_1/\sin A$, $AM_3=b\sin\omega_1\cot A$,

$AM_2=b\sin\omega_1/\sin A$, $AM_1=c\sin\omega_1\cot A$.

$\therefore$ $AM.AM_3=AM_1.AM_2$ or $AM:AM_2=AM_1:AM_3$.

$\therefore$ M, M_1, M_2, M_3 are concyclic.

Similarly for the remaining pairs of triangles.



(4) Let t, t_1, t_2 be the tangents from D, C, A to the Brocard circle.

Then $t=(DI.Dk_3)^{\frac12}=\frac{1}{\sqrt{2}}b\sin\frac12 C\cos\frac12 C$,

$$t_1=(CP.CI)^{\frac{1}{2}}=\frac{1}{\sqrt{2}}b\cos C,\ t_2=(AP.AI)^{\frac{1}{2}}=\frac{1}{\sqrt{2}}b\sin C.$$

$\therefore\ t^2=t_1t_2.\quad \therefore\ t_1:t=t:t_2.$

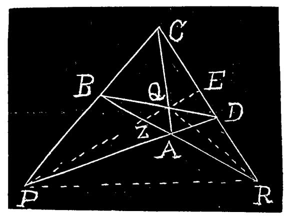
Similarly for the other triangles.

Also proved by *J. R. HITT*.

188. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

ABCD is a quadrilateral whose diagonal triangle is *PQR*, *P* on *AD* and *R* on *AB*. *PQ* meets *AB* in *Z*. If *C* moves along *PB* what will happen to *Z*?

Solution by J. R. HITT, A. M., Coronal Institute, San Marcos, Tex., and G. W. GREENWOOD, A. M., McKendree College, Lebanon, Ill.



Let the figure be drawn as indicated in the proposition. Now, the sides *PQ*, *PR*, of the diagonal triangle, and the opposite sides *AD*, *BC*, of the quadrilateral, form a harmonic pencil. The three rays, *PR*, *PD*, *PC*, remain fixed as *C* moves along *BC*; hence, ray *PE* remains fixed. Therefore, *Z* is fixed, since *RB* is fixed.

Also demonstrated by *G. B. M. ZERR*.

189. Proposed by J. C. CORBIN, Pine Bluff, Ark.

The perpendicular from the right angle on the hypotenuse of a right-angled-triangle is a harmonic mean between the segments of the hypotenuse made by the point of contact of the inscribed circle. [From Casey's *Sequel to Euclid.*]

Solution by LON C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, and J. E. SANDERS, Hackney, Ohio.

Denote by *ABC* the right-angled-triangle, whose sides opposite the angles A, B, C are a, b, c, respectively, where C is the right angle.

Let r be the radius of the incircle with center O, which is tangent to AB at the point G. Draw CD perpendicular to AB, OE to BC, and OF to AC. Then by the condition of the problem, we have

$$CD=\frac{2BG.GA}{BG+GA}\ldots.(1).$$

Since $AG=AF=b-r=\frac{1}{2}[c-(a-b)]$, and $BG=BE=a-r=\frac{1}{2}[c+(a-b)]$, the right member of (1) reduces to

$$\frac{c^2-(a-b)^2}{2c}=\frac{ab}{c}\ldots.(2).$$

Now from the similar right triangles BDC and BCA, we have

$$CD:AC=BC:AB, \text{ or } CD:a=b:c,$$

from which $CD=ab/c$, which agrees with result (2).

Also solved by *J. R. HITT*, *J. SCHEFFER*, and *G. B. M. ZERR*.

CALCULUS.

144. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find volume common to the two solids

$$[x/a]^{\frac{2}{3}}+[y/b]^{\frac{2}{3}}=[z/c]^{\frac{2}{3}}, \quad [y/b]^{\frac{2}{3}}+[z/c]^{\frac{3}{8}}=[x/a]^{\frac{1}{3}}.$$

Solution by the PROPOSER.

From the equations $[x/a]^{\frac{2}{3}}+[y/b]^{\frac{2}{3}}=[z/c]^{\frac{2}{3}}$ and $[y/b]^{\frac{2}{3}}+[z/c]^{\frac{2}{3}}=[x/a)^{\frac{1}{3}}$ we find the limits of z to be $z=c\{[x/a]^{\frac{2}{3}}+[y/b]^{\frac{2}{3}}\}^{\frac{3}{2}}$ to $z=c\{[x/a]^{\frac{1}{3}}-[y/b]^{\frac{2}{3}}\}^{\frac{3}{2}}$. Eliminating z we get

$$y=\pm\frac{b}{\sqrt{8}}\left[\frac{x}{a}\right]^{\frac{1}{2}}\{1-[x/a]^{\frac{1}{3}}\}^{\frac{3}{2}}=y'.$$

The limits of x are $x=0$ to $x=a$.

$$\therefore\ V=2c\int_0^a\int_0^{y'}[\{(x/a)^{\frac{1}{3}}-(y/b)^{\frac{2}{3}}\}^{\frac{3}{2}}-\{(x/a)^{\frac{2}{3}}+(y/b)^{\frac{2}{3}}\}^{\frac{3}{2}}]dxdy.$$

Let $x/a=u^3$, $y/b=v^3$.

$$\therefore\ V=18abc\int_0^1\int_0^{\sqrt{[\frac{1}{2}u(1-u)]}}u^2v^2\{(u-v^2)^{\frac{3}{2}}-(u^2+v^2)^{\frac{3}{2}}\}dudv$$

$$=\tfrac{3}{8}abc\int_0^1\Big(u^2(2+u+2u^2)\sqrt{(1-u^2)}-8(1-u^2)^{\frac{5}{2}}+3u^2\sin^{-1}\left[\frac{1-u}{2}\right]$$

$$+6u^5\log\left[\frac{1+\sqrt{(1-u^2)}}{u}\right]\Big)u^2du$$

Let $u=\cos\theta$.

$$\therefore\ V=\tfrac{3}{16}abc\int_0^{\frac{1}{2}\pi}\Big(4\cos^2\theta\sin\theta+2\cos^3\theta\sin\theta+4\cos^4\theta\sin\theta-16\sin^5\theta+3\theta\cos^2\theta$$

$$+12\cos^5\theta\log\left[\frac{1+\sin\theta}{\cos\theta}\right]\Big)\cos^3\theta\sin\theta d\theta=\frac{115\pi abc}{2048}.$$

148. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College Defiance, Ohio.

Helmholtz's differential equation for the strength of an electric current C at any time t, is $C=E/R-L/R\times dC/dt$. Solve this equation, supposing $C=0$ when $t=0$; and E, R, L are to be regarded as constants.

Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa., and J. SCHEFFER, A. M. Hagerstown, Md.

The equation can be written $LdC=(E-CR)dt$.

$$\therefore \frac{LdC}{E-CR}=dt, \text{ or } \frac{L}{R}\log(E-CR)+t+B=0.$$

When $C=0$, $t=0$, and $B=-\frac{L}{R}\log E$.

$$\therefore t+\frac{L}{R}\log(E-CR)=\frac{L}{R}\log E, \text{ or } t+\frac{L}{R}\log\left(\frac{E-CR}{E}\right)=0.$$

149. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College Philadelphia, Pa.

Find the volume contained between the plane $z=(a-x)\cot\beta$ and the surface $xz^2=(a-x)(x^2-y^2)$.

Solution by the PROPOSER.

The z limits are $z=\sqrt{\frac{a-x}{x}(x^2+y^2)}$ to $z=(a-x)\cot\beta$.

Eliminating z, $y=\pm\sqrt{[x(a\cot^2\beta-x\operatorname{cosec}^2\beta)]}=y'$.
The x limits are 0 and $a\cos^2\beta=x'$.

$$V=2\int_0^{x'}\int_0^{y'}\left[(a-x)\cot\beta-\sqrt{\frac{a-x}{x}(x^2+y^2)}\right]dxdy$$

$$=\int_0^{x'}\Bigg[\sqrt{x}\,(c-x)\cot\beta\sqrt{[a\cot^2\beta-x\operatorname{cosec}^2\beta]}$$

$$-x\sqrt{(ax-x^2)}\log\left(\frac{\sqrt{(a-v)}\cot\beta+\sqrt{(a\cot^2\beta-x\operatorname{cosec}^2\beta)}}{\sqrt{x}}\right)\Bigg]dx.$$

Let $x=a\cos^2\beta\sin^2\theta$.

$$\therefore\ V=2a^3\cos^3\beta\cot^2\beta\int_0^{\frac{1}{2}\pi}(1-\cos^2\beta\sin^2\theta)\sin^2\theta\cos^2\theta d\theta$$

$$-2a^3\cos^5\beta\int_0^{\frac{1}{2}\pi}\sqrt{(1-\cos^2\beta\sin^2\theta)}\sin^4\theta\cos\theta\log\left(\frac{\sqrt{(1-\cos^2\beta\sin^2\theta)}+\cos\theta}{\sin\beta\sin\theta}\right)d\theta.$$

Let $\cos\beta\sin\theta=\sin\varphi$; then the second term becomes

$$2a^3\int_0^{\frac{1}{2}\pi-\beta}\sin^4\varphi\cos^2\varphi\log\left[\frac{\cos\varphi\cos\beta+\sqrt{(\cos^2\beta-\sin^2\varphi)}}{\sin\beta\sin\varphi}\right]d\varphi=2a^3\int_0^{\frac{1}{2}\pi-\beta}(\tfrac{1}{16}\varphi-$$

$$\tfrac{1}{16}\sin\varphi\cos\varphi-\tfrac{1}{24}\sin^3\varphi\cos\varphi+\tfrac{1}{6}\sin^5\varphi\cos\varphi)\frac{\cos\beta d\varphi}{\sin\varphi\sqrt{(\cos^2\beta-\sin^2\varphi)}}.$$

$$\therefore\ V=\tfrac{1}{16}\pi a^3\cos^3\beta\cot^2\beta(1+\sin^2\beta)+\tfrac{1}{8}a^3\cos\beta\int_0^{\frac{1}{2}\pi}d\theta+\tfrac{1}{12}\cos^3\beta\int_0^{\frac{1}{2}\pi}\sin^2\theta d\theta$$

$$-\tfrac{1}{3}a^3\cos^5\beta\int_0^{\frac{1}{2}\pi}\sin^4\theta d\theta-\tfrac{1}{8}a^3\int_0^{\frac{1}{2}\pi}\frac{\sin^{-1}(\cos\beta\sin\theta)d\theta}{\sin\theta\sqrt{(1-\cos^2\beta\sin^2\theta)}}.$$

$$\therefore\ V=\tfrac{1}{16}\pi a^3\cot\beta\operatorname{cosec}\beta-\tfrac{1}{24}\pi a^3\cos^3\beta$$

$$-\tfrac{1}{8}a^3\int_0^{\frac{1}{2}\pi}\frac{\sin^{-1}(\cos\beta\sin\theta)d\theta}{\sin\theta\sqrt{(1-\cos^2\beta\sin^2\theta)}}.\quad \tfrac{1}{8}a^3\int_0^{\frac{1}{2}\pi}\frac{\sin^{-1}(\cos\beta\sin\theta)d\theta}{\sin\theta\sqrt{(1-\cos^2\beta\sin^2\theta)}}$$

$$=\tfrac{1}{8}a^3\cos\beta\int_0^{\frac{1}{2}\pi}(1+\tfrac{2}{3}\cos^2\beta\sin^2\theta+\tfrac{8}{15}\cos^4\beta\sin^4\theta+\tfrac{16}{35}\cos^6\beta\sin^6\theta+\ldots.)d\theta$$

$$=\tfrac{1}{16}\pi a^3(\cos\beta+\tfrac{1}{3}\cos^3\beta+\tfrac{1}{5}\cos^5\beta+\tfrac{1}{7}\cos^7\beta+\ldots.)=\tfrac{1}{16}\pi a^3\log\cot\tfrac{1}{2}\beta.$$

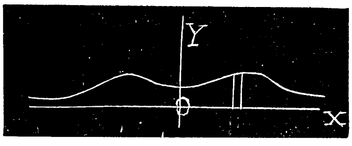
$$\therefore\ V=\tfrac{1}{48}\pi a^3(3\cot\beta\operatorname{cosec}\beta-2\cos^3\beta-3\log\cot\tfrac{1}{2}\beta).$$

150. Proposed by E. B. ESCOTT, Instructor in Mathematics, University of Michigan, Ann Arbor, Mich.

Find total area between the curve $x^4y-x^2+4y-1=0$ and the x-axis.

Solution by J. E. SANDERS, Hackney, Ohio, and the PROPOSER.

The equation may be written $y=\dfrac{x^2+1}{x^4+4}$.



$$\text{Area}=2\int_0^\infty ydx=2\int_0^\infty\frac{x^2+1}{x^4+4}dx$$

$$=\tfrac{1}{4}\left[\int_0^\infty\frac{x+2}{x^2-2x+2}dx-\int_0^\infty\frac{x-2}{x^2+2x+2}dx\right]=\tfrac{1}{8}\Big[\log(x^2-2x+2)$$

$$-\log(x^2+2x+2)+6[\tan^{-1}(x+1)+\tan^{-1}(x-1)]\Big]_0^\infty$$

$$=\tfrac{1}{8}\left[\log\frac{x^2-2x+2}{x^2+2x+2}+6\tan^{-1}\frac{2x}{2-x^2}\right]_0^\infty=\tfrac{1}{8}(6\pi)=\tfrac{3}{4}\pi.\quad \text{Answer.}$$

$\left[\tan^{-1}\frac{2x}{2-x^2}\right]_{x=\infty}=\tan^{-1}0=\pi$, since the integrand has increased and passed through ∞ for $x=\sqrt{2}$.

Also solved by *G. B. M. ZERR.*

MECHANICS.

144. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Pressure is applied perpendicularly to the plane surface yz, bounding an otherwise infinite isotropic solid. Find the resultant displacements, if the pressure varies as $\sin\left(\frac{2\pi y}{a}\right)+\sinh\left(\frac{2\pi y}{a}\right)$.

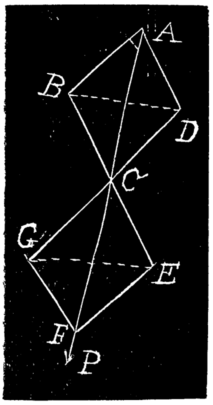
No solution of this problem has been received.

145. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

ABCD, *GCEF* are equal parallelograms, *DCG* and *BCE* being straight lines. If the figure be considered as formed of smooth light jointed bars and if *BD* be a light rod, and the whole be suspended from *A*, find the stress in *BD* if a weight be hung from *F*. Also find the stress if a light rod *GE* replace *BD*.

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Since the bars are light we can disregard their weight. Let P be the weight. Then by virtual work



$$Pd(AC)+Sd(BD)=0\ldots.(1).$$

But $AC^2+BD^2=2AD^2+2DC^2$.

$\therefore ACd(AC)+BDd(BD)=0\ldots.(2)$.

From (1) and (2),

$$\frac{d(AC)}{d(BD)}=-\frac{S}{P}=-\frac{BD}{AC}.$$

$\therefore S=\frac{P.BD}{AC}$. Similarly for GE.

$S_1=\frac{P.GE}{CF}$. The stress is the same for both.

146. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A diffraction grating, with lines .05 mm. apart is held in front of a Bunsen's burner in which common salt is volatilized, and, when viewed through a telescope it is found that the angular distances of the first, second, third, fourth, fifth, and sixth bright bands from the central one are respectively 41′, 1°21′, 2°2′, 2°42′, 3°23′ and 4°3′. Required the wave length of sodium light.

Solution by M. E. GRABER, Graduate Student, Heidelberg University. Tiffin, Ohio, and the PROPOSER.

Let λ=wave length, d=distance apart of grating, θ=angular distance, n=order from central one. Then $\lambda=d\sin\theta/n$.

$$\frac{.05\sin 41'}{1}=.000596305\text{mm.},\quad \frac{.05\sin 1^\circ 21'}{2}=.000588995\text{mm.},$$

$$\frac{.05\sin 2^\circ 2'}{3}=.000591348\text{mm.},\quad \frac{.05\sin 2^\circ 42'}{4}=.000588831\text{mm.},$$

$$\frac{.05\sin 3^\circ 21'}{5}=.000590160\text{mm.},\quad \frac{.05\sin 4^\circ 3'}{6}=.000588558\text{mm.}$$

λ=the mean of these six. $\therefore$ $\lambda=.0005906995$mm.

DIOPHANTINE ANALYSIS.

99. Proposed by the late HON. JOSIAH H. DRUMMOND, LL. D.

If p and q are such values of x and y as fulfill the conditions $x^2\pm y^2-1$=a square, find, in terms of p and q, the expression for an indefinite number of other values.

Solution by G. B. M. ZERR. A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$x^2\pm y^2=a^2+1=b$, or $\dfrac{x^2}{b}\pm\dfrac{y^2}{b}=1$.

Let $x=p$, and $y=q$ be a solution. Then $(p^2/b\pm q^2/b)^n=1$.

$\therefore$ $[x/\sqrt{b}+y\sqrt{(\pm1)}/b][x/\sqrt{b}-y\sqrt{(\pm1)}/\sqrt{b}]=[p/\sqrt{b}+q\sqrt{(\pm1)}/\sqrt{b}]^n$ $\times[p/\sqrt{b}-q\sqrt{(\pm1)}/\sqrt{b}]^n$.

Put $x/\sqrt{b}+y\sqrt{(\pm1)}/\sqrt{b}=[p/\sqrt{b}+q\sqrt{(\pm1)}/\sqrt{b}]^n$.

$x/\sqrt{b}-y\sqrt{(\pm1)}/\sqrt{b}=[p/\sqrt{b}-q\sqrt{(\pm1)}/\sqrt{b}]^n$.

$$\therefore x=\frac{\sqrt{b}}{2}\{[p/\sqrt{b}+q\sqrt{(\pm1)}/\sqrt{b}]^n+[p/\sqrt{b}-q\sqrt{(\pm1)}/\sqrt{b}]^n\},$$

$$y=\frac{\sqrt{b}}{2\sqrt{\pm1}}\{[p/\sqrt{b}+q\sqrt{(\pm1)}/\sqrt{b}]^n-[p/\sqrt{b}-q\sqrt{(\pm1)}/\sqrt{b}]^n\}.$$

By ascribing to n the values 1, 2, 3, ..., as many solutions as are desired can be obtained.

100. Proposed by A. H. BELL, Hillsboro, Ill.

Prove that every indeterminate equation of the second degree can be reduced to $x^2-Ay^2=Bz^2$. [*Legendre*.]

Solution by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University. Cal.

Let $ax'^2+2hx'y'+by'^2+2gx'+2fy'+c=0$ represent any determinate equa-

tion of the second degree, in which x' and y' are the two indeterminates, and a, b, c, f, g, h being integers.

Solving this equation as a quadratic in x', and transposing $hy'+g$, we have

$$ax'+hy'+g=\pm\sqrt{[(h^2-ab)y'^2+2(hg-af)y'+(g^2-ac)]}\ldots.(1).$$

In order that the values of x' and y' may be positive integers, the expression in (1) under the radical, which we may denote by $Ay'^2+2qy'+r$, must be a perfect square; that is,

$$Ay'^2+2qy'+r=t^2,\text{ suppose, }\ldots.(2).$$

Solving (2) as a quadratic in y', we have, after squaring each member,

$$(Ay'q+)^2=At^2+(q^2-Ar)\ldots.(3).$$

Now set $Ay'+q=u$, $q^2-Ar=B$, and transpose At^2, then (3) becomes

$$u^2-At^2=B\ldots.(4).$$

Since u and t may be fractional, assume $u=x/z$ and $t=y/z$, then substitute in (4), and we get $x^2-Ay^2=Bz^2$.

Also solved by the *PROPOSER*.

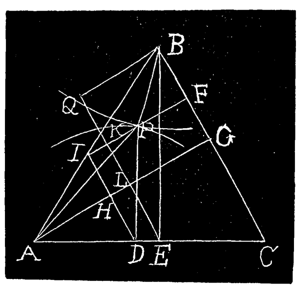
AVERAGE AND PROBABILITY.

120. Proposed by L. C. WALKER, A. M., Graduate Student. Leland Stanford Jr. University, Cal.

If a given ellipse be placed at random on another equal ellipse, find the chance that the center of the first will fall on the second.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let the one ellipse, center B, move parallel to itself so that it is always tangent to the other fixed ellipse, center A, at some point P. Let AC, BC be the directions of the axes-majores of the ellipses each semi-axes a, b. Let A be the origin, AC axis of abscissas; (x, y), coördinates of P; (m, n), coördinates of B; $\theta=\angle ACB$. Draw PF, AG perpendicular to BC; PD, BE perpendicular to AC; BQ parallel to AG; DI, EQ parallel to BC.



Then $AG=AL+BQ=m\sin\theta+n\cos\theta$,

$BG=QE-LE=n\sin\theta-m\cos\theta$.

$PF=AG-AH-IP=(m-x)\sin\theta+(n-y)\cos\theta$.

$BF=BG+HD-DI=(n-y)\sin\theta-(m-x)\cos\theta$.

$\therefore$ The equation to ellipse, center B, is

$$\frac{[(m-x)\cos\theta-(n-y)\sin\theta]^2}{a^2}+\frac{[(m-x)\sin\theta+(n-y)\cos\theta]^2}{b^2}=1....(1).$$

The equation to ellipse, center A, is

$$x^2/a^2+y^2/b^2=1....(2).$$

The invariant of these equations are

$$\triangle=\triangle_1=-a^4b^4.$$
$$\Theta=a^2b^2[b^2(m^2+n^2)+c(m\sin\theta+n\cos\theta)^2-c^2\sin^2\theta-3a^2b^2].$$
$$\Theta_1=a^2b^2[b^2m^2+a^2n^2-c^2\sin^2\theta-3a^2b^2), \text{ where } c=a^2-b^2.$$

Let $m=r\cos\varphi$, $n=r\sin\varphi$.

$$\therefore\ \Theta=a^2b^2\{r^2[a^2\sin^2(\theta+\varphi)+b^2\cos^2(\theta+\varphi)]-c^2\sin^2\theta-3a^2b^2\}$$
$$=a^2b^2(r^2A-c^2\sin^2\theta-3a^2b^2), \text{ suppose.}$$

$$\Theta_1=a^2b^2[r^2(a^2\sin^2\varphi+b^2\cos^2\varphi)-c^2\sin^2\theta-3a^2b^2]$$
$$=a^2b^2(r^2B-c^2\sin^2\theta-3a^2b^2), \text{ suppose.}$$

The condition that the ellipses should touch is

$$(\Theta\Theta_1-9\triangle\triangle_1)^2=4(\Theta^2-3\triangle\Theta_1)(\Theta_1^2-3\triangle_1\Theta).$$

The locus of B is

$$A^2B^2r^8-2(A+B)[(5a^2b^2+c^2\sin^2\theta)AB-2a^2b^2(A^2+B^2)]r^6$$
$$+[(A^2+B^2)(c^4\sin^4\theta-6a^2b^2c^2\sin^2\theta-27a^4b^4)+2AB(2c^4\sin^4\theta$$
$$+12a^2b^2c^2\sin^2\theta+27a^4b^4)]r^4-2c^4\sin^4\theta(A+B)(3a^2b^2+c^2\sin^2\theta)r^2$$
$$+c^6\sin^6\theta(4a^2b^2+c^2\sin^2\theta)=0.$$

This equation represents two loci, one when the moving ellipse is tangent externally, the other when the moving ellipse is tangent internally.

If α is the average area of the former, the chance is $p=\pi ab/\alpha$.

121. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find the average area of the pentagon formed by joining five random points on the surface of a given circle.

Solution by the PROPOSER.

Let a=radius of given circle, A=its area, $\triangle$=required average, $\triangle'$=average area when the five points are taken on both the circle A and a concentric annulus B, $\triangle_1$=average area when four points are taken on A and one on B.

Then $(\triangle'+\triangle)A=5B(\triangle_1-\triangle)$. But $\triangle:\triangle'=A:A+B$.

$\therefore\ \triangle'=(A+B)\triangle/A$. $\therefore\ \triangle=\frac{5}{6}\triangle_1$.

Let one point O be on the circumference of the circle and the other four points P, Q, R, S anywhere on its surface. Let $OP=u$, $OS=x$, $OR=y$, $OQ=z$, $\angle AOE=\theta$, $\angle BOE=\varphi$, $\angle COE=\psi$, $\angle DOE=\delta$. Then $OA=2a\sin\theta=u'$, $OB=2a\sin\varphi=x'$, $OC=2a\sin\psi=y'$, $OD=2a\sin\delta=z'$.

Area $OPQRS=\frac{1}{2}uz\sin(\theta-\delta)+\frac{1}{2}yz\sin(\delta-\psi)+\frac{1}{2}xy\sin(\psi-\varphi)=C$.

$$\therefore\ \triangle=\tfrac{5}{6}\triangle_1=\frac{5\int_0^\pi\int_0^\theta\int_0^\delta\int_0^\psi\int_0^{u'}\int_0^{x'}\int_0^{y'}\int_0^{z'}Cuxyz\,d\theta\,d\delta\,d\psi\,d\varphi\,du\,dx\,dy\,dz}{6\int_0^\pi\int_0^\theta\int_0^\delta\int_0^\psi\int_0^{u'}\int_0^{x'}\int_0^{y'}\int_0^{z'}uxyz\,d\theta\,d\delta\,d\psi\,d\varphi\,du\,dx\,dy\,dz}$$

$$=\frac{20}{\pi^4a^8}\int_0^\pi\int_0^\theta\int_0^\delta\int_0^\psi\int_0^{u'}\int_0^{x'}\int_0^{y'}\int_0^{z'}Cuxyz\,d\theta\,d\delta\,d\psi\,d\varphi\,du\,dx\,dy\,dz$$

$$=\frac{20}{3\pi^4a^6}\int_0^\pi\int_0^\theta\int_0^\delta\int_0^\psi\int_0^{u'}\int_0^{x'}\int_0^{y'}[4au\sin(\theta-\delta)\sin\delta+4ay\sin(\delta-\psi)\sin\delta$$

$$+3xy\sin(\psi-\varphi)]uxy\sin^2\delta\,d\theta\,d\delta\,d\psi\,d\varphi\,du\,dx\,dy$$

$$=\frac{160}{9\pi^4a^3}\int_0^\pi\int_0^\theta\int_0^\delta\int_0^\psi\int_0^{u'}\int_0^{x'}[3u\sin(\theta-\delta)\sin\delta+3x\sin(\psi-\varphi)\sin\psi$$

$$+4a\sin(\delta-\psi)\sin\delta\sin\psi]ux\sin^2\delta\sin^2\psi\,d\theta\,d\delta\,d\psi\,d\varphi\,du\,dx$$

$$=\frac{320}{9\pi^4a}\int_0^\pi\int_0^\theta\int_0^\delta\int_0^\psi\int_0^{u'}[3u\sin(\theta-\delta)\sin\delta+4a\sin(\psi-\varphi)\sin\psi\sin\varphi$$

$$+4a\sin(\delta-\psi)\sin\delta\sin\psi]u\sin^2\delta\sin^2\psi\sin^2\varphi\,d\theta\,d\delta\,d\psi\,d\varphi\,du$$

$$=\frac{2560a^2}{9\pi^4}\int_0^\pi\int_0^\theta\int_0^\delta\int_0^\psi[\sin(\theta-\delta)\sin\theta\sin\delta+\sin(\delta-\psi)\sin\delta\sin\psi$$

$$+\sin(\psi-\varphi)\sin\psi\sin\varphi]\sin^2\theta\sin^2\delta\sin^2\psi\sin^2\varphi\,d\theta\,d\delta\,d\psi\,d\varphi$$

$$=\frac{320a^2}{9\pi^4}\int_0^\pi\int_0^\theta\int_0^\delta[4\sin(\theta-\delta)\sin\theta\sin\delta(\psi-\sin\psi\cos\psi)+3\sin^2\psi$$

$$+4\sin(\delta-\psi)\sin\delta\sin\psi(\psi-\sin\psi\cos\psi)-\sin^4\psi-3\psi\sin\psi\cos\psi]\sin^2\theta\sin^2\delta\sin^2\psi\,d\theta\,d\delta\,d\psi$$

$$=\frac{10a^2}{27\pi^4}\int_0^\pi\int_0^\theta[105\delta-72\delta^2\sin\delta\cos\delta+84\delta\sin^2\delta-120\delta\sin^4\delta-105\sin\delta\cos\delta$$

$$-82\sin^3\delta\cos\delta+32\sin^5\delta\cos\delta+96\sin(\theta-\delta)\sin\theta\sin\delta(\delta-\sin\delta\cos\delta)^2]\sin^2\theta\sin^2\delta d\theta d\delta$$

$$=\frac{5a^2}{162\pi^4}\int_0^\pi(360\theta^2-144\theta^3\sin\theta\cos\theta-360\theta^2\sin^4\theta+252\theta^2\sin^2\theta+336\theta\sin^5\theta\cos\theta$$

$$-132\theta\sin^3\cos\theta-615\theta\sin\theta\cos\theta+72\sin^8\theta-250\sin^6\theta-301\sin^4\theta+255\sin^2\theta)\sin^2\theta d\theta$$

$$=\frac{5a^2}{12\pi}\left(5-\frac{1001}{144\pi^2}\right).$$

MISCELLANEOUS.

116. Proposed by J. A. CALDERHEAD, B.Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

Prove that

$$-\left|\alpha^2\beta^2\gamma^2\right|^2\begin{vmatrix}a&b&c\\b&d&e\\c&e&f\end{vmatrix}^2=\left\|\begin{matrix}\begin{vmatrix}a&b&c&\alpha_1\\b&d&e&\alpha_2\\c&e&f&\alpha_3\\\alpha_1&\alpha_2&\alpha_3&0\end{vmatrix}&\begin{vmatrix}a&b&c&\alpha_1\\b&d&e&\alpha_2\\c&e&f&\alpha_3\\\beta_1&\beta_2&\beta_3&0\end{vmatrix}&\begin{vmatrix}a&b&c&\alpha_1\\b&d&e&\alpha_2\\c&e&f&\alpha_3\\\gamma_1&\gamma_2&\gamma_3&0\end{vmatrix}\\\begin{vmatrix}a&b&c&\beta_1\\b&d&e&\beta_2\\c&e&f&\beta_3\\\alpha_1&\alpha_2&\alpha_3&0\end{vmatrix}&\begin{vmatrix}a&b&c&\beta_1\\b&d&e&\beta_2\\c&e&f&\beta_3\\\beta_1&\beta_2&\beta_3&0\end{vmatrix}&\begin{vmatrix}a&b&c&\beta_1\\b&d&e&\beta_2\\c&e&f&\beta_3\\\gamma_1&\gamma_2&\gamma_3&0\end{vmatrix}\\\begin{vmatrix}a&b&c&\gamma_1\\b&d&e&\gamma_2\\c&e&f&\gamma_3\\\alpha_1&\alpha_2&\alpha_3&0\end{vmatrix}&\begin{vmatrix}a&b&c&\gamma_1\\b&d&e&\gamma_2\\c&e&f&\gamma_3\\\beta_1&\beta_2&\beta_3&0\end{vmatrix}&\begin{vmatrix}a&b&c&\gamma_1\\a&b&c&\gamma_2\\c&e&f&\gamma_3\\\gamma_1&\gamma_2&\gamma_3&0\end{vmatrix}\end{matrix}\right\|$$

[From Muir's *Determinants*.]

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let x, y, z be the minors with respect to α_1, α_2, α_3 in the first row, u, v, w the minors with respect to β_1, β_2, β_3 in the second row, and r, s, t the minors with respect to γ_1, γ_2, γ_3 in the third row of the determinant $\triangle$ on the right hand side of the equality. Then

$$=-\begin{vmatrix}\alpha_1x-\alpha_2y+\alpha_3z, & \beta_1x-\beta_2y+\beta_3z, & \gamma_1x-\gamma_2y+\gamma_3z\\ \alpha_1u-\alpha_2v+\alpha_3w, & \beta_1u-\beta_2v+\beta_3w, & \gamma_1u-\gamma_2v+\gamma_3w\\ \alpha_1r-\alpha_2s+\alpha_3t, & \beta_1r-\beta_2s+\beta_3t, & \gamma_1r-\gamma_2s+\gamma_3t\end{vmatrix}$$

$$=\begin{vmatrix}\alpha_1&\alpha_2&\alpha_3\\\beta_2&\beta_2&\beta_3\\\gamma_1&\gamma_2&\gamma_3\end{vmatrix}\begin{vmatrix}x&y&z\\u&v&w\\r&r&t\end{vmatrix}=(\alpha_1\beta_2\gamma_3)\begin{vmatrix}x&y&z\\u&v&w\\r&s&t\end{vmatrix}$$

$$=(\alpha_1\beta_2\gamma_3)\begin{vmatrix} \begin{vmatrix} b & c & \alpha_1 \\ d & e & \alpha_2 \\ e & f & \alpha_3 \end{vmatrix} & \begin{vmatrix} a & c & \alpha_1 \\ b & e & \alpha_2 \\ c & f & \alpha_3 \end{vmatrix} & \begin{vmatrix} a & b & \alpha_1 \\ b & d & \alpha_2 \\ c & e & \alpha_3 \end{vmatrix} \\ \begin{vmatrix} b & c & \beta_1 \\ d & e & \beta_2 \\ e & f & \beta_3 \end{vmatrix} & \begin{vmatrix} a & c & \beta_1 \\ b & e & \beta_2 \\ c & f & \beta_3 \end{vmatrix} & \begin{vmatrix} a & b & \beta_1 \\ b & d & \beta_2 \\ c & e & \beta_3 \end{vmatrix} \\ \begin{vmatrix} b & c & \gamma_1 \\ d & e & \gamma_2 \\ e & f & \gamma_3 \end{vmatrix} & \begin{vmatrix} a & c & \gamma_1 \\ b & e & \gamma_2 \\ c & f & \gamma_3 \end{vmatrix} & \begin{vmatrix} a & b & \gamma_1 \\ b & d & \gamma_2 \\ c & e & \gamma_3 \end{vmatrix} \end{vmatrix}$$

Let A, B, C, etc., be the minors with respect to a, b, c, etc. Then

$$\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}^2 = \begin{vmatrix} A & -B & C \\ -B & D & -E \\ C & -E & F \end{vmatrix}$$

$$\therefore\ \triangle=(\alpha_1\beta_2\gamma_3)\begin{vmatrix} \alpha_1A-\alpha_2B+\alpha_3C, & \alpha_1B-\alpha_2D+\alpha_3E, & \alpha_1C-\alpha_2E+\alpha_3F \\ \beta_1A-\beta_2B+\beta_3C, & \beta_1B-\beta_2D+\beta_3E, & \beta_1C-\beta_2E+\beta_3F \\ \gamma_1A-\gamma_2B+\gamma_3C, & \gamma_1B-\gamma_2D+\gamma_3E, & \gamma_1C-\gamma_2E+\gamma_3F \end{vmatrix}$$

$$=-(\alpha_1\beta_2\gamma_3)^2\begin{vmatrix} A & -B & C \\ -B & D & -E \\ C & -E & F \end{vmatrix}=-(\alpha_1\beta_2\gamma_3)^2\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}^2.$$

117. Proposed by **W. J. GREENSTREET, M. A.**, Editor of The Mathematical Gazette, Stroud, England.

If $x\cos\alpha+y\cos\alpha=a\cos\theta+b\cos\varphi$, and $x\sin\alpha+b\sin\varphi=y\sin\alpha+a\sin\theta=\kappa$, find the maximum value of κ, and the values of x and y.

Solution by **G. B. M. ZERR, A.M., Ph. D.**, Professor of Chemistry and Physics, The Temple College. Philadelphia, Pa.

$$(x+y)\cos\alpha=a\cos\theta+b\cos\varphi\ldots.(1).$$
$$(x+y)\sin\alpha+a\sin\theta+b\sin\varphi=2\kappa\ldots.(2).$$

(1) in (2) gives

$$\frac{(a\cos\theta+b\cos\varphi)\sin\alpha}{\cos\alpha}+a\sin\theta+b\sin\varphi=2\kappa, \text{ or}$$

$$a\sin(\theta+\alpha)+b\sin(\varphi+\alpha)-2\kappa\cos\alpha=0=u.$$

$\therefore\ du/d\theta=a\cos(\theta+\alpha)=0,\ du/d\varphi=b\cos(\varphi+\alpha)=0.$

$\therefore\ \theta=\varphi=\frac{1}{2}\pi-\alpha$ for a maximum.

$\therefore\ \kappa=\frac{1}{2}(a+b)\sec\alpha$ is the maximum value.

$$x\cos\alpha+\frac{(\kappa-a\sin\theta)\cos\alpha}{\sin\alpha}=a\cos\theta+b\cos\varphi.$$

$$\therefore\ x=\frac{a\sin(\theta+\alpha)+b\cos\phi\sin\alpha-\kappa\cos\alpha}{\sin\alpha\cos\alpha}.$$

$$y=\cos\alpha+\frac{(\kappa-b\sin\phi)\cos\alpha}{\sin\alpha}=a\cos\theta+b\cos\phi.$$

$$\therefore\ y=\frac{b\sin(\phi+\alpha)+a\cos\theta\sin\alpha-\kappa\cos\alpha}{\sin\alpha\cos\alpha}.$$

When $\theta=\phi=\frac{1}{2}\pi-\alpha$ and $\kappa=\frac{1}{2}(a+b)\sec\alpha$,

$$x=\frac{a-b\cos2\alpha}{\sin2\alpha},\ y=\frac{b-a\cos2\alpha}{\sin2\alpha}.$$

118. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Show how to determine the illumination at any point of the surface of the water at the bottom of a deep well, due to the light from the sky.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let I=illumination, dQ the quantity of light that falls upon a small area dA of an illuminated surface, coming from an element of any bright surface dS.

Let O be the center of the element of the luminous surface, C the center of the illuminated area dA, and let $OC=r$, depth of well$=a$. Let θ=inclination of OC to normal at O, ϕ=inclination of OC to normal at C. Now if dA subtend solid angle $d\omega$ at O, and μ be the intrinsic brightness of dS, then

$$dQ=\mu dS\cos\theta d\omega,\ \text{but}\ d\omega=\frac{dA\cos\phi}{r^2}.$$

$\therefore\ dQ=\mu dSdA\cos\theta\cos\phi/r^2$.

$\therefore\ I=dQ/dA=\mu\cos\phi\cos\theta dS/r^2$.

If $d\sigma$ be the solid angle at C subtended by dS, then $d\sigma=\cos\theta dS/r^2$.

$\therefore\ I=\mu\cos\phi d\sigma$.

The solid angle at C cannot be larger than the top of the well will admit. Let b=the area of the top of well, then we may write $I=\mu b/a^2$ for center of surface of water, since $\theta=\phi=0$, and $S/r^2=b/a^2$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

170. Proposed by S. F. NORRIS, Professor of Astronomy and Mathematics, Baltimore City College, Baltimore, Md.

Find by strictly quadratic methods at least one set of values of x and y in the equations $x^2y^2+x=38$ and $xy+y^2=15$.

171. Proposed by IDA M. SCHOTTENFELTZ, A. M.

$ay^2+a=bxy+cx$, $bx^2+b=axy+cy$. Solve for x and y.

172. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Without solving the algebraically solvable quintic $y^5+py^3+\frac{1}{5}p^2y+r=0$, prove that it is irreducible in the domain of rationality (p, r).

GEOMETRY.

193. Proposed by PROFESSOR BEYENS.

Si le rapport du segment d'une base de la sphère à l'hemisphère est m/n, le rapport de l'hauteur du segment à deux bases qui resultera au rayon est égal à $2\sin\frac{1}{3}[\sin^{-1}(n-m)/n]$. [Problem 9699, *Educational Times.*]

194. Proposed by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

Glass paper weights, having the form of a regular tetrahedron, are to be packed for shipment, each in a paper box. Wanted to know the size and shape of the smallest box for the purpose. How much empty space in each box?

CALCULUS.

160. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A dog at the vertex of a right conical hill pursues a fox at the foot of the hill. How far will the dog run to catch the fox, if the dog runs directly toward the fox at all times, and the fox is continually running around the hill at its foot, the velocity of the dog being 6 feet per second, the velocity of the fox being 5 feet per second, the hill being 100 feet high and 200 feet in diameter at the base?

MECHANICS.

150. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

O is a point in the plane of a triangle, ABC, and D, E, F are the mid-points of the sides. Show, geometrically, that the system of forces OA, OB, OC is equivalent to the system OD, OE, OF.

DIOPHANTINE ANALYSIS.

109. Proposed by HARRY S. VANDIVER, Bala, Pa.

If $m+n+1$ is a prime integer, show that $m!\times n!\ -(-1)^{\frac{1}{2}(3m-n)}$ is divisible by $m+n+1$. For instance, $6!\times 4!-(-1)^7$ is divisible by 11.

110. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Prove the results stated in the foot-note at the end of §4 in the article by L. E. Dickson in the October number of the MONTHLY.

111. Proposed by HARRY S. VANDIVER, Bala, Pa.

Show that: If a rational, integral polynomial of the nth degree in x, becomes a prime for more than n values of x, then it cannot be resolved into rational factors. (Test with the expression $x^4+2x^3+x^2+2x+1$.)

AVERAGE AND PROBABILITY.

135. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

If the line joining two points taken at random in the surface of a given circle be the diagonal of a square, the chance that the square lies wholly within the circle is $2-4/\pi$.

NOTES.

Prof. Lon C. Walker is doing postgraduate work in Leland Stanford Jr. University. F.

Prof. Harry S. Vandiver is taking a course of mathematics in the University of Pennsylvania. F

On December 15, the University of Klausenberg will celebrate the hundredth anniversary of the birth of Johann Bolyai. F.

Dr. F. Schottky of Marburg has been named to succeed the late L. Fuchs as Professor in the University of Berlin. Dr. K. Hensel of Berlin has been named Professor in the University of Marburg. D.

Dr. Joseph Swain, President of the University of Indiana since 1893, and formerly Professor of Mathematics in Indiana and Stanford Universities, was installed as President of Swarthmore College on November 15. D.

In the reconstitution of the University of London, complete courses of study in the various faculties of the University College have been established.

Mention may be made of the endowment of the department of pure mathematics by Mr Astor. The University is to be no longer merely an examining board.

D.

The address, November 13, of Dr. E. W. Hobson, retiring president of the London Mathematical Society, will be on the "The infinite and the infinitesimal in mathematical analysis." At this meeting the triennial De Morgan medal is to be presented to A. G. Greenhill for his investigations in pure and applied mathematics. D.

The centenary of the birth of Abel, at Christiania, September 4-7, was regarded as a national event in Norway. The King of Norway and Sweden and his son Prince Eugen made a special journey from Stockholm. Among the mathematicians making addresses were H. Weber, Volterra, L. Sylow, Forsyth, Picard, Schwarz, Zeuthen, Henzel, and Mittag-Leffler. By a special act of the Storthing, power had been granted to the University of Christiania to confer honorary degrees on this special occasion. Among the twenty-nine mathematicians elected *Doctores Mathematicae* were Lord Kelvin, Lord Rayleigh, Dr. Salmon, Sir George Stokes, Prof. G. A. Darwin, Prof. A. R. Forsyth, Prof. Simon Newcomb, and Prof. J. Willard Gibbs. D.

Since the last issue of the MONTHLY two more of its valued friends and contributors have received their summons to close their earthly toil. Prof. P. H. Philbrick and Hon. Josiah H. Drummond will no more give us personal encouragement in our work, nor gladden the hearts of those who loved to study their contributions. While they are dead, yet they will long live in the memories of many of the readers of the MONTHLY. We give below a brief sketch of the life of Prof. Philbrick and in the next issue will appear a brief biography of Hon. Josiah H. Drummond, whose death occurred on Saturday, October, 25.

Philetus Harry Philbrick was born at Machias, New York, March 8, 1839 and died at Bedford, Oregon, October 10, 1902. His death was not expected and was due to heart failure in the early stages of a mild typhoid infection.

After the age of ten years, he had but a few weeks schooling, until at twenty, when he sent himself to the Tafton Collegiate Seminary at Tafton, (now Bloomington) Wisconsin. Here he attended school until 1862, when he enlisted in Company A, 20th Wisconsin Regiment and served until the end of the war. After returning from the war, he taught school for a few months at Prairie du Chein, Wisconsin, and then entered the University of Michigan 1866. He graduated from that institution in the class of 1868, having in two years completed courses entitling him to the degrees of C. E. and B. Sc. For five years until 1873, he was engaged in various constructions in Michigan. He then received the appointment as professor of civil engineering in the University of Iowa and held this position until 1887 when he resigned to enter upon active field work in southern Louisiana, with headquarters at Lake Charles. This remained his home until within a few weeks of his death. Because of advancing years and chronic malarial disease, he removed to southern Oregon where he anticipated great en-

joyment in his remaining years looking after a fruit farm, solving problems in various departments of mathematics and writing articles on mathematcial, scientific, and other subjects.

During his residence in Louisiana, he was chief engineer of the Kansas, Watkins & Gulf Railroad; Lake Charles & Gulf Railroad; North American Land and Timber Co. etc.

Professor Philbrick was a contributor to various mathematical and scientific journals, including *The Analyst*, *The Mathematical Magazine*, *The Mathematical Visitor*, *The Engineering News*, and *The Mathematical Monthly*.

In each of the mathematical journals mentioned above, appear a great many of his excellent solutions of problems of various kinds. In the *Mathematical Magazine* is a brief article of his in which he developes a new method for finding the superior limes of a root of an algebraic equation. His method has considerable merit, and is, in general, superior to the common method. In the *Mathematical Magazine* also appears an article of his "On the Abuse of Logarithms." This elicited a reply by Prof. Herbert A. Howe of the University of Denver, Denver, Colorado, and occasioned a long friendly controversey.

In 1901 Professor Philbrick wrote a Field Manual for Engineers. This work which was published by John Wiley & Sons, will take its place among the best works on that subject for many years to come. It is well written and has received the highest commendation from prominent civil engineers.

Professor Philbrick was a man of keen intellect. He was entirely frank and fearless in the expression of his views. He wrote many controversial articles championing the Scientific Education, Metric System, Spelling Reform, Liberal Religion, etc. Three grown children mourn the loss of the kindest of fathers. F.

BOOKS AND PERIODICALS.

The Foundations of Geometry. By David Hilbert. Authorized Translation by E. J. Townsend, Ph, D., University of Illinois. Chicago: The Open Court Publishing Co. 1902. Pages, vii+132.

Readers of the American Mathematical Monthly, may consult a technical review of this translation in *Science*, Vol. XVI. No. 399. Aug. 22, 1902. pp. 307-8, where in the interest of merest justice are pointed out some few among the blemishes in what Professor Townsend puts forth as a translation of Hilbert's beautiful 'Festschrift.' These blemishes are the more indefensible because Professor Townsend had before him, in addition to the limped original, the admirable French translation of L. Langel.

For example, Hilbert, so studiously sparing of words, uses the word *Erklarung* nine times on his first thirteen pages.

Townsend never renders it at all. Where he adds from Langel, he seems to have no better luck with his French than with his German. For example, p. 25, "This axiom gives us nothing directly concerning the existence of limiting points or of the idea of converg-

ence" is how he renders, 'Cet axiome ne nous dit rien sur l' existence de points limites ni sur la notion de convergence.'

On p. 125, the translation reads: "We easily see that the criterion of theorem 44 is fulfilled, and, consequently, it follows that every regular polygon can be constructed by the drawing of straight lines and the laying off of segments." From this we should suppose that Professor Townsend studied his geometry from the popular treatise of Mr. Wentworth between 1877 and 1887, which during those years contained on p. 224, proposition XIII § 387:

"To inscribe a regular polygon of any number of sides in a given circle."

GEORGE BRUCE HALSTED.

The Book We Need. By Leon Stefflre, LL. B., of Bowdle, S. Dak. 8vo. Cloth. 218 pages. Price $1.00. San Francisco: The Whitaker & Ray Co.

The title of this little book does not in the least suggest the nature of the subject matter. There are many books that are urgently needed by most of us and perhaps this is one that is needed by some of us. The book is really an elementary arithmetic in which are introduced a few deviations from customary usage. In the first place, the author uses the Greek letter delta, inverted, instead of the decimal point.

The work closes with an epilogue in which the author advances a number of novel ideas.

The printing is not very good as there is too much uniformity in type throughout.

B. F. F.

The Business Man's Arithmetic. By Prof. J. S. Hunter. 8vo. Stiff Paper Back. 71 pages. Price 25cts. San Francisco: The Whitaker & Ray Co.

In this little pamphlet, the author has attempted to develop a system of computation, simple, brief, and sufficiently comprehensive as to be applicable to any kind of problem in any kind of business, and which presents no elaborate rules to be memorized by the learner. The author has simply made cancellation the basis of his system. This principle, I presume, is used already by nine-tenths of the practical computers of the country. Even this book in the hands of one unacquainted with fundamental principles would accomplish very little. A book to be put in the hands of pupils should always present principles and not rules or "systems." B. F. F.

Elementary Arithmetic of the Octimal Notation. By Geo. H. Cooper. 8vo. Stiff Paper Back. 70 pages. Price, 25cts. San Francisco: The Whitaker & Ray Co.

The object of this little work seems to be to revolutionize the present decimal system of notation. The author says, "Any attempt to perpetuate the use of the decimal system is nothing short of a crime against humanity, since it fails in every department." It has long been observed that some other than the decimal system would be more convenient, for example, the duo-decimal system in which 12 is the radix. But to change a system which has grown up with the race, and which answers so admirably the purposes which it is to serve, as does the decimal system of notation, seems to me exceedingly futile. Would it not be far better to spend our energies in advocating a universal adoption of the decimal system of weights and measures, rather than waste our powers in trying to establish an entirely new system? B. F. F.

Differential and Integral Calculus. By Virgil Snyder, Ph. D. and John I. Hutchinson, Ph. D., of Cornell University. 8vo. Cloth. xvi+320 pages. Price, $2.50. New York and Chicago: The American Book Co.

This book compares very favorably with the others of the Cornell Mathematical Series. The part of the work on Differential Calculus is based largely upon McMahon & Snyder's Differential Calculus while the part on the Integral Calculus is entirely new.

The exercises are new and are carefully graded. Numerous illustrative examples are worked out and accompanied by helpful suggestions. The Derivative is presented vigorously as a limit. The treatment throughout the book is simple, clear, practical, and thoroughly rigorous, and in the hands of a live instructor will accomplish great good.

B. F. F.

The School Vistior An Elemenary Monthly Journal, Devoted to Difficult Work in Common School Studies. Price, $1.00 per year in advance. Published bp John S. Royer & Sons, Columbus, O.

The School Visitor was started in 1880, by Professor Royer, and its publication continued for 15 years, at the end of which time, owing to the nervous strain it caused and the tax it levied on the vital force of its editor, it was discontinued until 1900.

It is now appearing with all the vigor of its earlier days, and we trust that it may long continue to cultivate the minds and gladden the hearts of thousands of teachers who avail themselves of its influence.

The School Visitor is the most practical and stimulating periodical that the ordinary teacher can read. During the first year of its publication, it received contributions in mathematics from Prof. E. B. Seitz, Dr. Artemas Martin, Prof. Henry Gunder, and Dr. William Hoover, and the contributions of these gentlemen were sources of inspiration at that time to the writer. The *Visitor* in its mathematical department has at present the support of our valued contributor, Dr. G. B. M. Zerr, who solves every difficult problem that appears in it.

B. F. F.

The American Journal of Mathematics. Edited by Frank Morley and others. Published quarterly, under the auspices of Johns Hopkins University. Price, $5.00 per year.

The October number contains the following articles: On systems of Linear Differential Equations of the First Order, by Maxime Bocher; On the Quarternary Linear Homogenious Group and the Ternary Fractional Group, by T. M. Putnam; On Cardinal Numbers by A. N. Whitehead; On a Method of Constructing all the Groups of Order p^m, by G. A. Miller; Non-Euclidean Properties of Plane Cubics and of their First and Second Polars, by Henry Freeman Stecker.

B. F. F.

Annals of Mathematics. Published quarterly, under the auspices of Harvard University. Price, $2.00 per year in advance..

The October number cantains the following articles: The Geodesic Lines on an Anchor Ring, by Dr. G. A. Bliss; Proof of a Theorem concerning Isosceles Triangles, by Prof. H. F. Blichfeldt; An Elementary Exposition of Frobenius's Theory of Group-Characters and Group-Determinants, by Dr. L. E. Dickson; Communication concerning Mr. Ransom's Mechanical Construction of Conics, by Dr. E. V. Huntington.

B. F. F.

Periodicals Received: *The American Monthly Review of Reviews; The Literary Digest; Popular Astronomy*, November Number; *Monthly Weather Review; Scientific American; Mathematical Gazette; The Mathematical Messenger; The University Herald; The Ohio Educational Monthly; The Onio Teacher; Le Matematiche; Periodico di Matematica; School Science; The School Visitor; The Open Court.*

ERRATA.

Page 247, middle, in η_6' the term $+\rho_2\eta_5$ is omitted.

In solution of problem 112, Miscellaneous, "$A-C+B-D$" should read $A-C-B+D$, and the corresponding corrections throughout.

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THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. IX. DECEMBER, 1902. NO. 12.

CLOSED LOXODROMICS OF THE TORUS.

By PROFESSOR ARNOLD EMCH, University of Colorado.

1. In Vol. VI (1899), pp. 136-139,* of this journal, I have proved the following theorem:†

If the ratio R/r of a torus is a rational fraction, and if a loxodromic of the torus winds m times around the axis (z-axis) and n times around the axial circle ($x^2+y^2=R^2$), then every loxodromic intersecting the first orthogonally closes also and their numbers m_1 and n_1 are related to the corresponding numbers m and n of the first by the equation

$$\frac{n}{m}\cdot\frac{n_1}{m_1}=\frac{R^2-r^2}{r^2},$$

where R is the radius of the axial circle and r that of a meridian-section of the torus, and $R>r$.

If the equatorial plane (plane of the axial circle) be assumed as the xy-plane, u as the angle which the plane of a meridian (circle with center C and

* Slight corrections appear on p. 188, where a superfluous = occurs. ED. D.

† A torus, anchor-ring, is a surface of revolution generated by a circle which rotates about a fixed axis of its plane. A loxodromic on the torus is a curve which intersects all parallels (and consequently all meridians) at constant angles.

A clear discussion of the torus and its conformal representation may be found in F. Klein's little book: "Ueber Riemann s Theorie der Algebraischen Funktionen und ihrer Integrale." Leipzig, 1882, pp. 50-55.

radius r) makes with xz-plane, and v as the angle which the radius PC from C to a point P of this meridian makes with the perpendicular to the xy-plane, the rectangular coördinates of P may be expressed by

$$x=(R+r\sin v)\cos u,$$
$$y=(R+r\sin v)\sin u,$$
$$z=r\cos v,$$

and the (u, v)-equation of a closed loxodromic of the prescribed kind by

$$\sin\frac{n}{m}u=\frac{r+R\sin v}{R+r\sin v}^{*}\ldots.(1).$$

In this note I shall discuss the nature of these curves.

2. For this purpose substitute in (1) for $\sin u$ and $\sin v$ their equivalent expressions

$$\sin u=\frac{y}{\sqrt{(x^2+y^2)}},\quad \sin v=\frac{\sqrt{(x^2+y^2)}-R}{r}.$$

In this manner the Cartesian equation of the projection of the closed loxodromic on the (x, y)-plane is obtained. As n and m are positive integers, $\sin\frac{n}{m}u$ may be expressed algebraically in terms of $\sin u$. Hence, (1) becomes an algebraic expression in x and y, and we have the theorem:

The projection of a closed loxodromic of the torus on the xy-plane is an algebraic curve.

As the torus is a surface of the fourth order it follows further:

The loxodromic itself is algebraic.

3. I shall consider in particular the case where the loxodromic turns only once about the z-axis, *i. e.*, where $m=1$. Now

$$\sin nu=a_1\sin u+a_3\sin^3u+\ldots.+a_n\sin^n u,\ \ldots.(2)$$

when $n=2k+1$ is odd; and

$$\sin nu=\cos u(b_1\sin u+b_3\sin^3u+\ldots.+b_n\sin^{n-1}u),\ \ldots.(3)$$

when $n=2k$ is even.†

In the first case, $n=2k+1$, we obtain after setting $\sin u=\frac{y}{\sqrt{(x^2+y^2)}}$,

$$\sin(2k+1)u=\frac{y}{\sqrt{(x^2+y^2)}}\left[a_1+a_3\frac{y^2}{x^2+y^2}+\ldots.\ a_n\frac{y^{2k}}{(x^2+y^2)^k}\right],$$

* A. Emch: Ueber orthogonale Systeme und einige technische Anvendungen, Biel, 1898.

† See Jordan: Cours d'Analyse, Vol. 1, p. 238.

$$\text{or, } \sin(2k+1)u = \frac{y}{\sqrt{(x^2+y^2)}\,(x^2+y^2)^k} f(x, y), \quad \ldots.(4)$$

where $f(x, y)$ is a polynomial in x and y of degree $2k$. Substituting (4) and the value of $\sin v$ in (1) we get

$$\frac{ryf(x, y)}{(x^2+y^2)^k} = r^2 - R^2 + R\sqrt{(x^2+y^2)},$$

or when rationalized and rearranged:

$$[ryf(x, y) + (R^2-r^2)(x^2+y^2)^k]^2 - R^2(x^2+y^2)^{2k+1} = 0, \quad \ldots.(5)$$

which, clearly, is a polynomial of degree $4k+2=2n$.

In the second case, $n=2k$, we have

$$\sin 2k = \frac{xy}{(x^2+y^2)}\left[b_1 + b_3\frac{y^2}{x^2+y^2} + \ldots. + b_n\frac{y^{2k-2}}{(x^2+y^2)^{k-1}}\right], \quad \ldots.(6)$$

$$\text{or, } \sin 2k = \frac{xy}{(x^2+y^2)^k} g(x, y), \quad \ldots.(7)$$

where $g(x, y)$ is a polynomial of degree $2k-2$. Substituting again in (1) and proceeding similarly as when we obtained (5), we get

$$[rxyg(x, y) - R(x^2+y^2)^k]^2 - (r^2-R^2)(x^2+y^2)^{2k-1} = 0, \quad \ldots.(8)$$

which, clearly, is a polynomial of degree $4k=2n$. We have therefore the theorem:

The xy-projection of a closed loxodromic winding n times around the axial circle is an algebraic curve of order 2n. The loxodromic itself is either of order 4n or else is a part of a curve of this order.

4. As an example take $n=1$, so that (5) becomes

$$R^2(x^2+y^2) = (R^2-r^2+ry)^2$$

This represents an ellipse and the corresponding loxodromic is a circle (see *loc. cit.*), and it must therefore be cut out by a double tangent-plane of the torus. This result may be stated in the theorem:

The double-tangent-planes of a torus cut the torus in circles which intersect the parallels and meridians of the torus under constant angles.

The cylinder $R^2(x^2+y^2)=(R^2-r^2+xy)^2$, z=arbitrary, cuts the torus in two circles of this kind, which are equally inclined to the xy-plane, thus making together a degenerated curve of the fourth order. The remainder of the intersection is an imaginary curve of the fourth order (two imaginary conics).

Taking $R=2r$, we get $\frac{n}{m}\frac{n_1}{m_1}=3$, hence $n_1=3$. The orthogonal loxodromic

winds 3 times around the axial circle. Its equation is easily found to be

$$R^2(x^2+y^2)^3=[(R^2-r^2)(x^2+y^2)+r(3x^2y-y^3)]^2,$$

and is, as we expect, of the 6th order.

Every closed loxodromic has an n-fold symmetry with respect to meridian-planes. Hence the cylinder with its xy-projection as a base cuts the torus in another loxodromic which is the reflection of the first on the xy-plane. The loxodromic is thus generally of order $4n$.

The University of Colorado, November, 1902.

ON BALL'S HISTORY OF MATHEMATICS.

By DR. G. A. MILLER.

Among the few histories of mathematics in the English language the third edition of Ball's work is the latest and most extensive.* It is therefore natural that this work should find a place in the libraries of many teachers of mathematics in these days of deep interest in the history of science. The following corrections and additions may possibly prove helpful especially to those who do not have the time or opportunity to study Cantor's great work.†

On page 4, Ball says, "The Egyptians and Greeks simplified the problem by reducing a fraction to the sum of several fractions, in each of which the numerator was unity, so that they had to consider only various denominators: the sole exceptions being the fractions $\frac{2}{3}$ and $\frac{3}{4}$. This remained the Greek practice until the sixth century of our era."

Being unable to find any instance where the ancient Egyptians had used the fraction $\frac{3}{4}$ I recently wrote Ball about the matter and received the following reply, "I cannot find my authority (if I had one) for the statement that the Egyptians used the fraction $\frac{3}{4}$. I fear it must have been an error which I have repeated in each edition without verification."

The above quotation from Ball's History is also apt to convey a false impression in regard to the Greek methods of dealing with fractions. While it is true that the Greeks employed unit fractions to a considerable extent, yet they also employed fractions with a general numerator and had a general notation for such fractions. In these respects they differed very widely from the Egyptians, although one might be led to infer the contrary not only from the above quotation but also from the statement on page 76.

On page 13 it is stated that the history of mathematics written by

* A Short Account of the History of Mathematics. By W. W. R. Ball, The Macmillan Company, New York, 1901, pp. xxiv+527.

† Vorlesungen ueber Geschichte der Mathematik. Von Moritz Cantor, Teubner, Leipzig, 1884-1901.

Theophrastus has been lost. Although Cantor makes a similar remark on page 108 (Vol. 1) of his history, yet it seems to have been proved quite recently that such a history never existed.* The statement on page 59 that Books VII, VIII, IX, and X of Euclid's Elements are devoted to the theory of numbers is found in each of the three editions. Since Book X is devoted to irrational magnitudes it cannot be classed with theory of numbers according to the common use of this term.

As Archimedes had employed the formula to find the sum of an arithmetical progression and the Pythagoreans had found the sum of certain arithmetical series, the statement on page 87, "Hypsicles developed the theory of arithmetical progressions which had been so strangely neglected by the earlier mathematicians," seems to give too much credit to Hypsicles. Even the ancient Egyptians were acquainted with the formula to find the sum of an arithmetical series, while such series were also known among the Babylonians.

Unfortunately, Ball gives a number of dates as definitely established which are known only approximately. For instance, it is not known exactly when the author of the most influential Greek arithmetic lived, yet Ball states definitely on page 97 that Nicomachus was a Jew, who was born at Gerasa in the year 50. Two pages later he states that Ptolemy died in 168 which is equally uncertain. The famous Neopolitan lad of 16 who is mentioned on page 103 was Annibale Giordano and not Oltajano. His home was in Ottajano.

Among the references on page 125 there should be included the valuable work of Conant, entitled Number Concept, published by *The Macmillan Company*. The tribe of West Africa, mentioned on page 127, counted by multiples of six and not by multiples of seven. On the same page it is stated that the Hindoos used the abacus or swan-pan, while on page 161 we read more correctly that "the Arabs (like the Hindoos) seem also to have made little or no use of the abacus." The fact is that the existence of the abacus among the Hindoos has not yet been established.

In the Roman abaci which I have seen pictured elsewhere, the marginal grooves or wires were used for fractions whose denominators are 12, 24, 36, 48, and 72, and not for those whose denominators are 4, as stated by Ball on page 129. The object which Leonardo had in view in writing his famous Liber Abaci seems to have been entirely misstated on page 174. That the words "in order that the Latin race might no longer be deficient in that knowledge" cannot refer to the Arabic system of notation follows directly from Leonardo's own words. It is quite probable that they refer to the method of "false assumption" upon which so mnch stress is laid in the Liber Abaci. On page 184 we read that Bradwardine "was the first European to introduce the cotangent into trigonometry." This function had been used earlier by Robertus Anglicus.

Pages 185 and 186 present a very remarkable state of affairs. While Oresme is the greatest French mathematician of the fourteenth century, yet Ball

* Cf. Enestroem, *Bibliotheca Mathematica*, Vol. 3, 1902, p. 246. At this place Enestroem notes a number of errors in Ball's History. Some of the most important one are repeated in the present note.

says "I do not propose to discuss his writings." The Latitudines formarum of Oresme explains how to draw a curve which represents the changes of a function during a given period, just like our modern temperature curves, and exhibits some of the properties of such curves. It indicates a very important step towards analytic geometry and hence it is of great historical interest. It was studied very extensively and exerted a powerful influence on the mathematical teachings of those times. Why such a work should be passed in silence is difficult to see. This becomes the more remarkable if it is observed that Ball gives on page 186 the courses in mathematics offered at the University of Vienna in 1389. One of these courses was devoted to Latitudines but as he failed to explain this subject at its proper place he is compelled to give an incomplete list of these courses. He seems to have substituted "measurement of superficies" for latitudines formarum, which conveys a totally wrong impression.

On page 195 we read, "and the test of the accuracy of the result by casting out the nines was invented by the Arabs." This test seems to have been employed earlier by the Hindoos. Record is believed to have been the first to use the modern sign for equality and gives as his reason for selecting this particular symbol that no two things can be more equal than two parallel straight lines. Notwithstanding this definite statement by Record we are told on page 221 that the symbol $=$ was a recognized abbreviation for the word *est* in medieval manuscripts; and this would seem to indicate a more probable origin than Record's own words.

We shall call attention to only one more statement, which is not incorrect but somewhat misleading. In speaking about Gauss, on page 462 we are told that "he discussed the binomial equation of the form $x^n=1$: this involves the celebrated theorem that it is possible to construct by elementary geometry regular polygons of which the number of sides is $2^m(2^n+1)$, where m and n are integers and 2^n+1 is a prime."

Why should it be $2^m(2^n+1)$ when it might just as well be $3.2^m(2^n+1)$, $5.2^m(2^n+1)$, or $15.2^m(2^n+1)$, since regular polygons of which the number of sides is of any one of these forms can be constructed in a similar way when $n=0$ or $n>2$. In fact these cases when $n=0$ were all known at the time of Euclid while Gauss made the remarkable discovery that n can have any value greater than 2 provided 2^n+1 is prime. Before the time of Gauss not a single polygon for $n>2$ had been constructed by elementary methods.

It may be added that this matter is stated in an unsatisfactory manner in at least two other recent histories of mathematics. From the statement in Cajori's History of Elementary Mathematics page 74 the reader would be likely to infer that it was impossible to construct a regular polygon of 51 sides by elementary methods, while the English editor of Fink's History of Mathematics, page 207, would seem to imply that even the regular pentagon could not be constructed in this way.

These remarks are not intended to throw discredit on the works cited. Few books are accurate in all details and histories of mathematics are no excep-

tion to this rule. In fact from the great variety of subjects treated, the task of their authors is an unusually difficult one. It is therefore important that the reader should be on his guard and utilize all available material instead of relying completely on some one author.

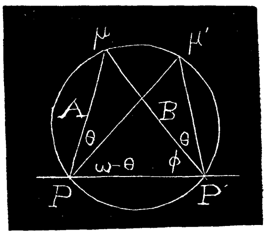
Stanford University, November, 1902.

A DEVELOPMENT OF THE CONIC SECTIONS BY KINEMATIC METHODS.

By JOHN JAMES QUINN, Ph. B., Head of the Department of Mathematics and Manual Training, Warren High School, Warren, Pa.

The Circle.

Proposition I. *If two lines A and B, pivoted at P and P′, respectively, be placed at any angle μ to each other, and both revolve in the same direction with the same angular velocity, the locus of their intersection is a circle.*



Let ϕ and ψ denote the angles which A and B in their initial position make with the line PP'. Then

$$\omega+\phi+\mu=180^\circ=\pi.$$

In any new position of A and B,

$$\omega-\theta+\phi+\theta+\mu'=\pi.$$

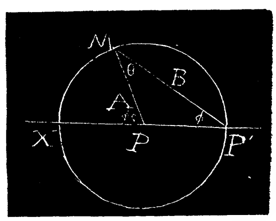
Hence $\mu=\mu'=$constant, so that the locus is a circle.

Remark. If p denotes the distance PP' between the pivots and C denotes the area of the generated circle, then

$$C=\pi p^2,\text{ if }\mu=30^\circ;\quad C=\tfrac{1}{2}\pi p^2,\text{ if }\mu=45^\circ;$$

$$C=\tfrac{1}{3}\pi p^2,\text{ if }\mu=60^\circ;\quad C=\tfrac{1}{4}\pi p^2,\text{ if }\mu=90^\circ.$$

Proposition II. *If two lines A and B, pivoted in an axis X, and initially coincident with it, revolve in the same direction, the one having twice the angular velocity of the other, then the locus of their intersection is a circle.*

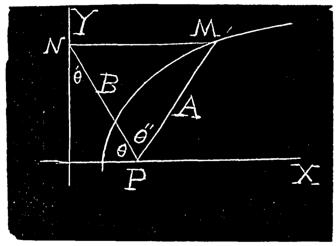


Let ϕ be the angle through which B moves in a unit of time; 2ϕ the angle through which A moves in the same time, and θ the vertical angle.

By the conditions, $2\phi=\phi+\theta$. Therefore $\theta=\phi$. Whence the triangle PMP' is isosceles. But the side PP' is constant. Therefore the side PM is constant. Therefore the locus of the intersection of A and B is a circle.

THE PARABOLA.

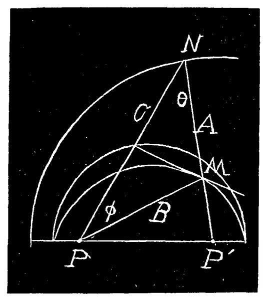
PROPOSITION III. *If two lines A and B, be pivoted in an axis X, at the same point P, and both revolve in the same direction, but the angular velocity of the line A is to the angular velocity of the line B as $2:1$, the locus of the intersection M of A with a line NM parallel to the axis through the point of intersection of B with a fixed perpendicular Y to the axis is a parabola.*



A revolves through 2θ; B revolves through θ. Since MN is parallel to X, $\theta=\theta'=\theta''$. Therefore $MP=MN$. Therefore the locus of M is a parabola.

THE ELLIPSE.

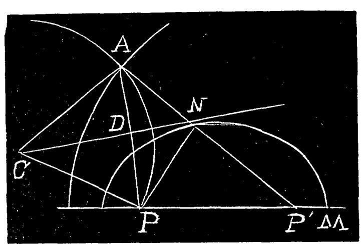
PROPOSITION IV. *If at two fixed points three lines A, B, and C, be pivoted, A at one point P', revolving at any velocity, and generating the directing circle with center P'; B and C at the other point P, revolving in the same direction, but the line C at such a rate that its intersection with A lies in the directing circle; and B at such a rate that the angle BC is constantly equal to the angle AC, then the locus of the intersection M of B and A is an ellipse.*



Let the angle $AC=\theta$, and the angle $BC=\phi$. Since the angle $\phi=$angle θ, the line $MN=$line PM. Therefore $PM+P'M=P'N+MN$. But $1'M+MN=A=$ a constant, being the radius of the directing circle. Therefore the locus of M is an ellipse.

COROLLARY. The perpendicular bisector of the line C is tangent to the ellipse at the point M.

PROPOSITION V. *If a circle of center C be described in the plane of an ellipse MN with its circumference passing through a focus P and cutting the directing circle in A and A', then the lines drawn from C to the mid-points D and D' of PA and PA' are tangent to the ellipse.*



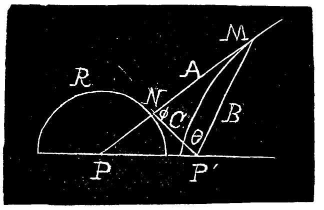
It is to be shown that CD and CD' (not drawn in the figure) are tangents to the ellipse with the foci P and P'. Let N be the intersection of $P'A$ with the ellipse. Then $P'N+NA=P'N+PN$, so that $NA=PN$. But CD is perpendicular to PA at its mid-point, since $CA=CP$. Hence CD passes through the vertex A. By the preceding corollary, CD is tangent to the ellipse at the point N. Similarly for the tangent CD'.

As a corollary, we derive the following construction for the tangents to an ellipse from any external point C in its plane: With C as a center, and CP as a radius, describe an arc cutting the directing circle in A and A'. Draw $P'A$ and $P'A'$ and let them intersect the ellipse at N and N'. Then CN and CN' are the required tangents.

A simple modification leads to a construction for a tangent to a parabola from any external point C. We have only to replace the directing circle by the directrix of the parabola.

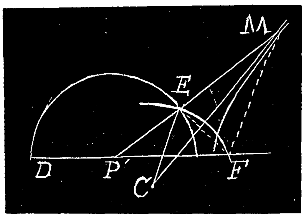
THE HYPERBOLA.

PROPOSITION VI. *If at two fixed points P and P', three lines A, B, and C, be pivoted, A at one point revolving in one direction at any velocity; B and C at the other pivot revolving in an opposite direction, C at such a rate that it constantly intersects A in the circumference of a directing circle described with P as a center, B at such a rate that the angle BC is constantly equal to the angle CA, then the locus of the intersection M of A and B is a hyperbola.*



Let the angle AC be denoted by ϕ and BC by θ. Since $\phi=\theta$, the segment NM=segment $P'M$ in any position. Therefore $PM-P'M=PM-NM=PN=$ constant. Therefore the locus of M is a hyperbola.

PROPOSITION VII. *If a circle with center C be described in the plane of a hyperbola passing through one focus P and intersecting the directing circle at E, and the other focal radius $P'M$ be drawn through this point E to meet the curve. at M, the line CM is tangent to the hyperbola.*



Draw MP and EP. The triangle EMP is isosceles and CM is perpendicular to the base PE at its mid-point A. Therefore it passes through the vertex M and is tangent to the hyperbola.

A METHOD FOR CONSTRUCTING AN HYPERBOLA, GIVEN THE ASYMPTOTES AND A FOCUS.

By ARCHIBALD HENDERSON, Ph. D., Associate Professor of Mathematics, University of North Carolina, Chapel Hill, N. C.

Consider any circle, whose center is the point $(0, y_0)$ and whose radius is the distance from this point to the focus $[\sqrt{(a^2+b^2)}, 0]$ of an hyperbola. The equation of this circle is

$$x^2+(y-y_0)^2=y_0^2+a^2+b^2,$$

$$\text{or } x^2+y^2-2y_0y-(a^2+b^2)=0\ldots.(1).$$

Now we may represent any point on an asymptote to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\ldots.(2)$$

by introducing the parameter t. Thus $(x_1, y_1)\equiv(at_1, bt_1)$ represents any point on the asymptote

$$y-\frac{b}{a}x=0\ldots.(3),$$

and $(x_2, y_2)\equiv(-at_2, bt_2)$ represents any point on the asymptote

$$y+\frac{b}{a}x=0\ldots.(4).$$

If the circle (1) cuts the asymptotes (3) and (4) in the specified points (x_1, y_1), (x_2, y_2), respectively, we have

$$(a^2+b^2)(t_1^2-1)=2by_0t_1\ldots.(5),$$
$$(a^2+b^2)(t_2^2-1)=2by_0t_2\ldots.(6).$$

By division we obtain

$$\frac{t_1^2-1}{t_2^2-1}=\frac{t_1}{t_2},$$

which may be written

$$(t_1-t_2)(t_1t_2+1)=0\ldots.(7).$$

The solution

$$t_1-t_2=0\ldots.(8)$$

shows that, for one position of (x_2, y_2), the line joining (x_1, y_1) and (x_2, y_2) is parallel to the x-axis. Discarding this case, let us consider the solution

$$t_1t_2+1=0\ldots.(9).$$

Since $(x_2, y_2)\equiv\left(\frac{a}{t_1}, \frac{-b}{t_1}\right)$, the equation of the line joining (x_1, y_1) and (x_2, y_2) is

$$\frac{y-bt_1}{\left(\frac{-b}{t_1}-bt_1\right)}=\frac{x-at_1}{\left(\frac{a}{t_1}-at_1\right)},$$

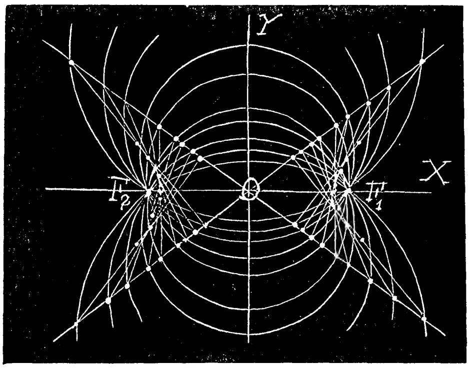
or

$$y=\frac{b}{a}\left(\frac{t_1^2+1}{t_1^2-1}\right)x-\frac{2bt_1}{t_1^2-1}\ldots.(10).$$

But this line touches the hyperbola (2), since

$$\frac{4b^2t_1^2}{(t_1^2-1)^2}\equiv b^2\left(\frac{t_1^2+1}{t_1^2-1}\right)^2-b^2 \ldots\ldots (11).$$

The lines joining the pairs of points (right hand, say) in which a system of co-axial circles, passing through the foci of an hyperbola, cuts the asymptotes, envelope that hyperbola.



Since, moreover, the middle point of the line joining (x_1, y_1), (x_2, y_2) lies on the hyperbola, we have the theorem:

The middle points of the lines joining the pairs of points in which a system of co-axial circles, passing through the foci of an hyperbola, cuts the asymptotes, describe that hyperbola.

These two theorems give two methods for constructing an hyperbola, the one by lines, the other by points, when the asymptotes and a focus are known.* Other constructions might readily have been given, but those given above seem the most instructive.

The University of Chicago, November, 1902.

* Compare the November number of the MONTHLY for a note by the writer on the converse of this problem.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

163. Proposed by CHRISTIAN HORNUNG, A.M., Professor of Mathematics, Heidelberg University, Tiffin, O.

Three Dutchmen and their wives went to market to buy hogs. The names of the men were Hans, Klaus, and Hendricks, and of the women, Gertrude, Anna, and Katrine; but it was not known which was the wife of each man. They each bought as many hogs as each man or woman paid shillings for each hog, and each man spent three guineas more than his wife. Hendricks bought 23 hogs more than Gertrude, and Klaus bought 11 more than Katrine. What was the name of each man's wife?

Solution by J. SCHEFFER, A. M., Hagerstown, Md., and M. E. GRABER, Heidelberg University, Tiffin, O.

Let x represent the number of one of the women's hogs, and y the number of *her* husband's; then by the conditions of the problem $y^2=x^2+63$. Consequently x^2+63 must be an integer, since $\sqrt{(x^2+63)}$ represents the number of hogs. The equation $y^2-x^2=63$ or $(y+x)(y-x)=63$ admits of three solutions, viz., 63×1, 21×3, and 9×7.

$$\therefore \left.\begin{array}{l} y+x=63 \\ y-x=1 \end{array}\right\} \text{ or } \left.\begin{array}{l} y+x=21 \\ y-x=3 \end{array}\right\} \text{ or } \begin{array}{l} y+x=9 \\ y-x=7; \end{array}$$

whence $y=32$, $x=31$; $y=12$, $x=9$; $y=8$, $x=1$. Since $32=9+23$ and $12=1+11$, 32 belongs to Hendricks, 12 to Klaus, 9 belongs to Katrine, 1 to Gertrude.

$\therefore$ 32 Hendricks, 31 Anna; 12 Klaus, 9 Katrine; 8 Hans, 1 Gertrude.

Hence, Hendricks is Anna's husband, Klaus is Katrine's husband, and Hans is Gertrude's husband.

Also solved by *W. R. LEBOLD*, *G. B. M. ZERR*, and *M. A. GRUBER*.

164. Proposed by JOSEPH V. COLLINS, Ph. D., Professor of Mathematics, State Normal School, Stevens Point, Wis.

Three women, the first with ten eggs, the second with thirty, and the third with fifty, went to market. They each got the same for their eggs, and all returned with the same money. What did they get?

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Let a, b, and c=the respective numbers of eggs the three women sold at y cents for every d eggs; and, for the remaining eggs, let x cents=price per egg.

Then $(10-a)x+\dfrac{a}{d}y=(30-b)x+\dfrac{b}{d}y=(50-c)x+\dfrac{c}{d}y$.

Whence, $y=\dfrac{b-a-20}{b-a}dx=\dfrac{c-a-40}{c-a}dx=\dfrac{c-b-20}{c-b}dx$.

Solving for a, b, and c, we find $a+c=2b$. For *positive* values, $a<9$, $b>20+a$ and <31, $c>40+a$ and <51.

Put $d=2$. Take $a=2$; then $b=24$ and 26, $c=46$ and 50, $y=2x/11$ and $\frac{1}{3}x$. For integral values, put $x=11$ and 3, respectively; then $y=2$ and 1. Therefore, 10 eggs brought $\frac{2}{2}\times 2c+8\times 11c=90c$, or $\frac{2}{2}\times 1c+8\times 3c=25c$; 30 eggs brought $\frac{24}{2}\times 2c+6\times 11c=90c$, or $\frac{26}{2}\times 1c+4\times 3c=25c$; and 50 eggs brought $\frac{46}{2}\times 2c+4\times 11c=90c$, or $\frac{50}{2}\times 1c=25c$.

Put $d=3$. Take $a=3$ and 6; then $b=24$ and 27, $c=45$ and 48, and $y=\frac{1}{7}x$. Put $x=7$, then $y=1$, and each of the women received 50c or 30c.

NOTE.—A special case of Mr. Gruber's solution is to let y=the price they received for the eggs per dozen and x=the price they received for the remaining eggs. Then $10x$=amount the first woman received, $2y+6x$=amount the second received, and $4y+2x$=amount the third received. Since they all received the same amount, we have $10x=2y+6x=4y+2x$. Therefore $y=2x$. Hence, if they sell them at 1, 2, 3, or 4c each, and 2, 4, 6, or 8c per dozen, they will receive the same sum.

Mr. Charles C. Cross and Mr. M. E. Graber solved the problem by assuming that they sell 7 eggs for a cent and the remaining eggs at 3 cents each. Thus each woman would get 10 cents.

Professor Zerr assumes that the first woman sells 1 egg for 1 cent and the remaining 9 at 6 cents each, receiving, therefore, 55 cents; the second sells 25 eggs for 1 cent each and 5 eggs at 6 cents each; and the third 49 eggs at 1 cent each and 1 egg for 6 cents. This way each would receive 55 cents. ED. F.

ALGEBRA.

161. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

If n quantities are made up of q sets of r each, find the number of permutations s at a time. It is supposed that the quantities in each set are alike, but different from those in the other sets.

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

If all different, the number of permutations$=n!$; but r things can be permuted in $r!$ ways, and q sets of r things in a set, can be permuted in $(r!)^q$ ways.

$\therefore s\times(r!)^q=n!$, or $s=\dfrac{n!}{(r!)^q}$.

162. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

If $x=\Sigma_0^\infty e^{-k[t+(2a\pi/h)]}\sin n\left(t+\dfrac{2a\pi}{h}\right)$, find value of x freed from Σ_0^∞.

Solution by the PROPOSER.

Let $2\pi/h=m$. $\therefore x=\Sigma_0^\infty e^{-k(t+am)}\sin n(t+am)$. a can have all positive integral values.

Let $C=\Sigma_0^\infty e^{-akm}\cos(anm)$, $S=\Sigma_0^\infty e^{-akm}\sin(anm)$.

Then $x=e^{-kt}(C\sin nt+S\cos nt)$. Now $C+S\sqrt{(-1)}=\Sigma_0^\infty e^{-am(k+n\sqrt{-1})}$

$$=\frac{1}{1-e^{-m(k+n\sqrt{-1})}}=\frac{1}{1-e^{-km}\cos(mn)-\sqrt{(-1)}^{-km}\sin(mn)}$$

$$=\frac{1-e^{-km}\cos(mn)+\sqrt{(-1)}e^{-km}\sin(mn)}{1-2e^{-km}\cos(mn)+e^{-2km}}.$$

$$\therefore C=\frac{1-e^{-km}\cos(mn)}{1-2e^{-km}\cos(mn)+e^{-2km}},\ S=\frac{e^{-km}\sin(mn)}{1-2e^{-km}\cos(mn)+e^{-2km}}.$$

$$\therefore x=\frac{e^{-kt}\{[1-e^{-km}\cos(mn)]\sin nt+e^{-km}\sin(mn)\cos nt\}}{1-2e^{-km}\cos(mn)+e^{-2km}}$$

$$=\frac{e^{-kt}-e^{-k(m+t)}\sin n(t-m)}{1-2e^{-km}\cos(mn)+e^{-2km}}.$$

NOTE ON PROBLEM 145 (UNSOLVED) BY H. S. VANDIVER, STUDENT, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PA.

It is possible to show that

$$F(a,\ b,\ c,\ d)=2b^2c^2+2c^2a^2+2a^2b^2+2a^2d^2+2b^2d^2+2c^2d^2-a^4-b^4-c^4-d^4$$

cannot be expressed as the product of two rational factors. For, assuming that we have

$$F(a,\ b,\ c,\ d)=f(a,\ b,\ c,\ d)f'(a,\ b,\ c,\ d)$$

(by symmetry both f and f' must contain all the letters a, b, c, and d). Put $a=b$, $c=d$. Then

$$F(a,\ a,\ d,\ d)=f(a,\ a,\ d,\ d)f'(a,\ a,\ d,\ d)=12a^2d^2-2a^4-2d^4.$$

That is, $12a^2d^2-2a^4-2d^4$ must be resoluble into two rational factors in a and d, since neither $f(a,\ a,\ d,\ d)$ nor $f'(a,\ a,\ d,\ d)$ can equal unity. It is evident however that $12a^2d^2-2a^4-2d^4$ does not possess this property.

GEOMETRY.

190. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Find the locus of the centers of sections of an ellipsoid by planes which are at a constant distance from the center.

Solution by the PROPOSER.

The center of the ellipsoid being the origin, and $(\alpha,\ \beta,\ \gamma)$ being the center of the section, its equation is found to be

$$\frac{\alpha}{a^2}x+\frac{\beta}{b^2}y+\frac{\gamma}{c^2}z-\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)=0\ldots.(1).$$

The perpendicular from the center of the ellipsoid upon it is

$$\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)\div\sqrt{\left(\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}+\frac{\gamma^2}{c^4}\right)}=k,$$

a constant, by the problem. This gives the required locus, which, by rationalizing, is easily seen to be a surface of the fourth degree.

Excellent solutions were also received from *PROFESSORS ZERR*, *WALKER*, and *SCHEFFER*.

CALCULUS.

151. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Integrate the differential equation, $xy\dfrac{\partial^2 z}{\partial x.\partial y}=bx\dfrac{\partial z}{\partial x}+ay.$

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Let $x=e^u$, $y=e^v$; then with $x\dfrac{d}{dx}=\theta$, $y\dfrac{d}{dy}=\theta'$, the given equation reduces to $\theta(\theta'-b)z=ae^v\ldots.(1)$, in which u and v are the independent variables.

The integral of (1) is

$$z=c\phi(y)+\frac{ay}{1-b}\log x+c_1\log x+c_2....(2),$$

noticing that $u=\log x$ and $v=\log y$.

Also solved by *G. B. M. ZERR*, *G. R. DEAN*, *L. C. WALKER*, and *J. SCHEFFER*.
Professor Walker should have been credited with a solution of Problem 148.

152. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Solve the differential equation, $e^x\left[\frac{dy}{dx}-y\log x\right]-a[\log x+1]=0$.

Solution by BEULAH FRAIZER, Sophomore Student, Missouri School of Mines, Rolla, Mo.

Dividing by e^x, $\frac{dy}{dx}-y\log x=ae^{-x}[\log x+1]$. The integrating factor is $e^{-\int\log x dx}=e^{+x(1-\log x)}=\frac{e^x}{x^x}$. Multiplying by this,

$$\frac{e^x}{x^x}\frac{dy}{dx}-\frac{y\log xe^x}{x^x}=a\frac{\log x+1}{x^x}y.$$

The left hand member is the derivative of $\frac{e^x}{x^x}y$. Hence we have

$$\frac{e^x}{x^x}y=a\int\frac{\log x+1}{x^x}dx=-ax^{-x}+c.$$

The solution is therefore, $y=\frac{cx^x-a}{e^x}$.

Also solved by *G. B. M. ZERR*, and *L. C. WALKER*.

153. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the equation of the loxodromic curve on an oblate spheroid.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Take the figure to Problem 95, Calculus, page 79, No. 3, Vol. VII. Let $AG=\theta$, $EP=x$, $CO=b$, $OB=a$, and $CG=$an elliptical arc; $\angle QPN=\beta$. Then

$$PN=xd\theta,\ QN=ds=\sqrt{\frac{a^2-e^2x^2}{a^2-x^2}}dx,\ \frac{PN}{QN}=\tan\beta.$$

$$\therefore\ d\theta=\frac{\tan\beta\sqrt{(a^2-e^2x^2)}dx}{x\sqrt{(a^2-x^2)}}.\quad \text{Let } x=\frac{a\cos\varphi}{\sqrt{(1-e^2\sin^2\varphi)}}.$$

$$\therefore\ d\theta=\frac{(1-e^2)\tan\beta d\varphi}{\cos\varphi\,(1-e^2\sin^2\varphi)}.\quad \text{Let } \sin\varphi=y.$$

$$\therefore d\theta=\frac{(1-e^2)\tan\beta dy}{(1-y^2)(1-e^2y^2)}.$$

$$\theta=\tfrac{1}{2}(1-e^2)\tan\beta\left(\log\frac{1+y}{1-y}+\frac{1}{e}\log\frac{1+ey}{1-ey}\right)+C.$$

$$\therefore \theta=\tfrac{1}{2}(1-e^2)\tan\beta\left(\log\frac{1+\sin\varphi}{1-\sin\varphi}+\frac{1}{e}\log\frac{1+e\sin\varphi}{1-e\sin\varphi}\right)+C$$

$$=(1-e^2)\tan\beta\left(\log\cdot\frac{\sqrt{(a^2-e^2x^2)}+\sqrt{(a^2-x^2)}}{x\sqrt{(1-e^2)}}\right.$$

$$\left.+\frac{1}{e}\log\frac{\sqrt{(a^2-e^2x^2)}+e\sqrt{(a^2-x^2)}}{a\sqrt{(1-e^2)}}\right)+C.$$

DIOPHANTINE ANALYSIS.

92. **Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.**

Find the sides of integral right triangles when the difference of the legs is given.

II. **Solution by the PROPOSER.**

From $(2mn)^2+(m^2-n^2)=(m^2+n^2)^2$, the general formula for prime integral right triangles, we find the difference of the legs to be m^2-n^2-2mn, or $(m-n)^2-2n^2$, the difference of a square and two times a square. This difference in prime integral right triangles is a prime number of the form $8p\pm1$, or a product or square of such prime numbers. Mathematically speaking, there is an indefinite number of right triangles for each difference of legs.

The first 20 differences of legs of prime right triangles are 1, 7, 17, 23, 31, 41, 47, 49 $(=7^2)$, 71, 73, 79, 89, 97, 103, 113, 119 $(=7\times17)$, 127, 137, 151, and 161.

When the difference of the legs is 2, 3, 4, 5, 6, 8, etc., the sides of the triangles are 2, 3, 4, 5, 6, 8, etc., times the sides of triangles in which the difference of the legs is 1. A difference of 14 is 14 times the difference of 1, and 2 times the difference of 7. And so on for all multiple right triangles.

Put $m^2-n^2-2mn=\pm d$. Then $m^2-2mn+n^2=2n^2\pm d$, and $m=n\pm\sqrt{(2n^2\pm d)}$. As already shown, d=the difference of a square and two times a square. Therefore, put $d=2r^2-s^2$. We will now have to find n in terms of r and s, so that $\sqrt{[2n^2\pm(2r^2-s^2)]}$ becomes rational. This is done by forming a series of convergents with r and s in accordance with the convergents of $\sqrt{2}$. Hence the following:

$$\begin{array}{rccccl}\sqrt{(2n^2\pm d)}=s, & 2r\pm s, & 4r\pm3s, & 10r\pm7s, & 24r\pm17s, & \text{etc.}\\ n=r, & r\pm s, & 3r\pm2s, & 7r\pm5s, & 17r\pm12s, & \text{etc.}\\ m=r\pm s, & 3r\pm2s, & 7r\pm5s, & 17r\pm12s, & 41r\pm29s, & \text{etc.}\end{array}$$

The *minus* sign indicates merely difference in these expressions, thus avoiding negative terms. In some of the calculations the differences are but a repetition of the sums.

To continue the series, the value of m is the next succeeding value of n, and the value of $m+n$ is the next succeeding value of $\sqrt{(2n^2 \pm d)}$.

Take $d=1=2\times1^2-1^2$. Then $r=1$ and $s=1$. Whence, $n=1$, 2, 5, 12, 29, etc.; $m=2$, 5, 12, 29, 70, etc.

Substituting these values in $(2mn)^2+(m^2-n^2)^2=(m^2+n^2)^2$, we find the following right triangles in which the difference of the legs is 1: 4, 3, 5; 20, 21, 29; 120, 119, 169; 696, 697, 985; 4060, 4059, 5741; etc.

Take $d=7=2\times2^2-1^2$. Then $r=2$ and $s=1$. Whence $n=2$, 1, 3, 4, 8, 9, 19, 22, etc.; $m=3$, 4, 8, 9, 19, 22, 46, 53, etc.

The right triangles are 12, 5, 13; 8, 15, 17; 48, 55, 73; 72, 65, 97; 304, 297, 425; 396, 403, 565; etc.

The sides of another set of triangles will be 7 times the sides of those the difference of whose legs is 1; as, 28, 21, 35; 140, 147, 203; 840, 833, 1183; etc.

REMARK ON PROBLEM 98, BY CHARLES C. CROSS, WHALEYVILLE, VA.

The solution given by Dr. Drummond of the second part of this problem does not appear to me to satisfy all the required conditions.

Let x and y be the numbers. Then

$$x+1=\square=a^2(\text{say})....(1);\quad y+1=\square=b^2(\text{say})....(2);$$
$$x+y+1=\square=c^2(\text{say})....(3);\quad x-y+1=\square=d^2(\text{say})....(4).$$

(1) and (2) in (3) and (4) give $a^2+b^2-1=c^2$ and $a^2-b^2+1=d^2$; adding $2a^2=c^2+d^2$. Let $c=m+n$ and $d=m-n$, then $a^2=m^2+n^2$. Let $a^2=(p^2+q^2)^2$, $m^2=(p^2-q^2)^2$, and $n^2=(2pq)^2$. Then $c=p^2-q^2+2pq$, and $d=p^2-q^2-2pq$. Therefore $x=(p^2+q^2)^2-1$, and $y=4pq(p^2-q^2)$.

In order that this value of y may satisfy the conditions of the problem, p^2-q^2 must equal $pq\pm1$. Whence $q=[\sqrt{(5p^2\pm4)}-p]/2$ in which $5p^2\pm4$ is to be made a square.

Let $p=2$, then $q=1$. $\therefore$ $x=24$, $y=24$.

Let $p=3$, then $q=2$. $\therefore$ $x=168$, $y=120$.

Whence $x+1=13^2$; $y+1=11^2$; $x+y+1=17^2$; and $x-y+1=7^2$. And so on for other values of p.

101. Proposed by HARRY S. VANDIVER, Bala, Pa.

Prove that it is impossible to find integral values for x, y, and z such that the relation $x^2y+xz^2=y^2z$ is satisfied.

Solution by the PROPOSER.

Suppose that x, y, and z are integers that satisfy $x^2y+xz^2=y^2z....(1)$, then we find

$$\begin{vmatrix} x & y & -z \\ y & -z & x \\ -z & x & y \end{vmatrix}^3 = \begin{vmatrix} X & Y & Z \\ Y & Z & X \\ Z & X & Y \end{vmatrix} \ldots.(2),$$

where
$$X=x^3+y^3-z^3-3xyz$$
$$Y=3(x^2y+xz^2-y^2z)=0.$$
$$Z=3(xy^2+yz^2-x^2z)$$

Then substituting in (2) and expanding

$$X^3+Z^3=(x^3+y^3-z^3+3xyz)^3$$

which is impossible, since the sum of two integral cubes cannot be an integral cube. (For a proof, see Euler's *Algebra.*) Hence the impossibility of (1) is established.

Also solved by the late *JOSIAH H. DRUMMOND.*

AVERAGE AND PROBABILITY.

122. Proposed by F. M. PRIEST, St. Louis, Mo.

Suppose each of the nine digits to be placed in a wheel, and five of them drawn at random therefrom, and written down in the order drawn. What is the probability the number thus expressed will be greater than 50,000?

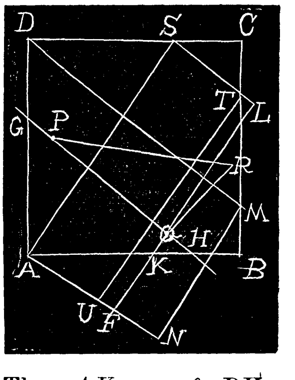
Solution by G. B. M. ZERR, A.M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and J. F. LAWRENCE, A. B., Breckenridge, Mo.

If 5, 6, 7, 8, or 9 be drawn first, the number will be greater than 50,000.

The chance of drawing 5 is $\frac{1}{9}$; of drawing 6, $\frac{1}{9}$; of drawing 7, $\frac{1}{9}$; of drawing 8, $\frac{1}{9}$; of drawing 9, $\frac{1}{9}$. The chance of drawing 5, 6, 7, 8, or 9 is, therefore, $\frac{5}{9}$. Therefore the chance that the number is greater than 50,000 is $\frac{5}{9}$.

123. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Three points are taken at random within a square. What is the probability that the angle formed by joining them is acute?



Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let $ABCD$ be the square; P, Q, R the three random points. Through P, Q draw GH, and through Q draw KL perpendicular to GH. When P is between G and Q the angle PQR will be obtuse if R lies on the opposite side of LK from P. Draw AF perpendicular to KL; AES perpendicular to GH; ST parallel to GH; MN and TU parallel to KL; DM parallel to GH.

Let $AB=a$, $PQ=x$, $AF=y$, $AE=z$, $\angle KAF=\theta$. Then $AK=y\sec\theta$, $BK=$

$a-y\sec\theta$, when L is on BC, $BL=(a-y\sec\theta)\cot\theta$, when L is on CD, and $CL=a(1-\tan\theta)-y\sec\theta$. Area $LBK=\frac{1}{2}(a-y\sec\theta)^2\cot\theta=\frac{1}{2}u$. Area $LCBK=\frac{1}{2}a(2a-2y\sec\theta-a\tan\theta)=\frac{1}{2}u_1$. An element of area at P is $xdxd\theta$; at Q, $dydz$. The limits of θ for $\frac{1}{2}u$ are $\frac{1}{4}\pi$ and $\frac{1}{2}\pi$, and 0 and $\frac{1}{4}\pi$; for $\frac{1}{2}u_1$, 0 and $\frac{1}{4}\pi$; of y for $\theta=\frac{1}{4}\pi$ to $\theta=\frac{1}{2}\pi$, $y=0$ to $y=a\cos\theta$; for $\theta=0$ to $\theta=\frac{1}{4}\pi$, $y=AN=a(\sec\theta-\sin\theta)=y_2$ to $y=a\cos\theta$, and $y=AU=a(1-\tan\theta)\sec\theta=y_3$ to $y=y_2$, and $y=a(\cos\theta-\sin\theta)=y_1$ to $y=y_3$, for $\frac{1}{2}u_1$, $y=0$ to $y=y_1$; of z, for $y=0$ to $a\cos\theta$, $z=y\tan\theta$ to $z=a\cos\theta$ and $z=a\cos\theta$ to $z=a\operatorname{cosec}\theta-y\cot\theta=z_1$; for $y=y_2$ to $y=a\cos\theta$, $z=y\tan\theta$ to $z=z_1$, for $y=y_3$ to $y=y_2$, $z=y\tan\theta$ to $z=a\cos\theta$ and $z=a\cos\theta$ to $z=z_1$, for $y=y_1$ to $y=y_3$, $z=y\tan\theta$ to $z=a\cos\theta$ and $z=a\cos\theta$ to $z=a\sec\theta$, for $\frac{1}{2}u_1$ the limits of z are $y\tan\theta$ to $a\cos\theta$ and $a\cos\theta$ to $a\sec\theta$; of x, 0 and $y+z\tan\theta=x_1$, and 0 and $y+a\operatorname{cosec}\theta-z\cot\theta=x_2$. The limits of x must be doubled for the case when P is on the opposite side of LK. The whole number of ways the three points can be taken is a^6. Let p' be the chance that PQR is obtuse.

$$p'=\frac{1}{a^6}\left[\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}\int_0^{a\cos\theta}ud\theta du+\int_0^{\frac{1}{4}\pi}\int_{y_3}^{y_2}ud\theta du\right]\left[\int_{y\tan\theta}^{a\cos\theta}\int_0^{x_1}dzxdx+\int_{a\cos\theta}^{z_1}\int_0^{x_2}dzxdx\right]$$

$$+\frac{1}{a^6}\left[\int_0^{\frac{1}{4}\pi}\int_{y_1}^{y_3}ud\theta du+\int_0^{\frac{1}{4}\pi}\int_0^{y_1}u_1d\theta du\right]\left[\int_{y\tan\theta}^{a\cos\theta}\int_0^{x_1}dzxdx+\int_{a\cos\theta}^{a\sec\theta}\int_0^{x_2}dzxdx\right]$$

$$+\frac{1}{a^6}\int_0^{\frac{1}{4}\pi}\int_{y_2}^{a\cos\theta}\int_{y\tan\theta}^{z_1}\int_0^{x_1}ud\theta dydzxdx=\frac{1}{6a^6}\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}\int_0^{a\cos\theta}[(y+a\sin\theta)^3\sec\theta\operatorname{cosec}\theta-$$

$$(y+a\sin\theta-a\cos\theta)^3\operatorname{cosec}^5\theta\sec\theta-y^3\sec^5\theta\operatorname{cosec}\theta]ud\theta dy$$

$$+\frac{1}{6a^6}\int_0^{\frac{1}{4}\pi}\left[\int_{y_3}^{y_2}[(y+a\sin\theta)^3\sec\theta\operatorname{cosec}\theta-y^3\sec^5\theta\operatorname{cosec}\theta-(y+a\sin\theta-a\cos\theta)^3\right.$$

$$\operatorname{cosec}^5\theta\sec\theta]udy$$

$$+\int_{y_1}^{y_3}[(y+a\sin\theta)^3\sec\theta\operatorname{cosec}\theta-y^3\sec^5\theta\operatorname{cosec}\theta-y^3\sin\theta\sec\theta]udy$$

$$+\int_0^{y_1}[(y+a\sin\theta)^3\sec\theta\operatorname{cosec}\theta-y^3\sec^5\theta\operatorname{cosec}\theta-y^3\sin\theta\sec\theta]u_1dy$$

$$\left.+\int_{y_2}^{a\cos\theta}(a^3\sec^2\theta\operatorname{cosec}\theta-y^3\sec^5\theta\operatorname{cosec}\theta)udy\right]d\theta$$

$$=\tfrac{1}{360}\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}(14\sin\theta\cos\theta+14\cos^2\theta+6\cot\theta+\cot^2\theta-\operatorname{cosec}^2\theta-20\cot\theta\operatorname{cosec}^2\theta$$

$$+45\cot^2\theta\operatorname{cosec}^2\theta-36\cot^3\theta\operatorname{cosec}^2\theta+10\cot^4\theta\operatorname{cosec}^2\theta)d\theta$$

$$+\tfrac{1}{360}\int_0^{\frac{1}{4}\pi}(4\sin^2\theta+2\cos^2\theta+42\sin\theta\cos\theta-12\sin\theta\cos^3\theta-20\sin^3\theta\cos\theta+\sec^8\mathrm{cosec}^2\theta$$

$$+6\sec^7\theta\,\mathrm{cosec}\theta-60\sec^2\theta\,\mathrm{cosec}\theta-6\sec\theta\,\mathrm{cosec}\theta-36\tan\theta+10\tan^2\theta$$

$$+20\tan^3\theta-\cot^2\theta+20\,\mathrm{cosec}^2\theta+123\sec^2\theta-50\tan\theta\sec^2\theta+5\tan^2\theta\sec^2\theta$$

$$+14\tan^3\theta\sec^2\theta-103\tan^4\theta\sec^2\theta+186\tan^5\theta\sec^2\theta-236\tan^6\theta\sec^2\theta$$

$$+192\tan^7\theta\sec^2\theta-60\tan^8\theta\sec^2\theta+36\tan^9\theta\sec^2\theta+84\tan^{10}\theta\sec^2\theta$$

$$+45\tan^{12}\theta\sec^2\theta)d\theta=\tfrac{119927}{400400}-\tfrac{11}{144}\log2.$$

$$3p'=\tfrac{359781}{400400}-\tfrac{11}{48}\log2,\ p=1-3p'=\tfrac{11}{48}\log2+\tfrac{40619}{400400}.\quad \therefore\ p=.260292.$$

124. Proposed by F. P. MATZ. Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Find the average area of a spherical polygon of $n=6$ sides.

No solution of this problem has been received.

MISCELLANEOUS.

118. Proposed by O. W. ANTHONY, New York, N. Y.

If f is determined by the equation $f(\mu\nu)=f(\mu)f^{-1}(\nu)+f(\nu)f^{-1}(\mu)$, where f^{-1} is the inverse of f, show that $f[(2)^\mu]=\frac{k^{\mu+1}}{2^{\mu+1}}$, where k is a constant.

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$f(\mu\nu)=ff^{-1}(\mu\nu)+ff^{-1}(\mu\nu)$, but $ff^{-1}=1$. $\therefore f(\mu\nu)=2(\mu\nu)$. $\therefore f=2$.

$\therefore (2)^\mu=(f)^\mu$ or $f[(2)^\mu]=(f)^{\mu+1}=(2)^{\mu+1}$.

$\therefore f[(2)^\mu]=(\frac{1}{2}k)^{\mu+1}$, where $k=4$.

119. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Show how to determine the illumination at any point of the surface of the water at the bottom of a deep well, due to the light from the sky.

A solution of this problem appeared in the November number. The problem was incorrectly numbered. ED.

120. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Prove $\Sigma\cos^4x-2\Pi\cos^2x+2\Pi\sin^2x=1-\sin(\Sigma)\sin\Pi(y+z-x)$.

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$$\sin(\Sigma)\sin\Pi(y+z-x)$$

$$=\sin(x+y+z)\sin(x+y-z)\sin(z-y+x)\sin(z+y-x)$$

$$=[\sin^2(x+y)-\sin^2 z][\sin^2 z-\sin^2(y-x)]$$
$$=\sin^2 z[\sin^2(x+y)+\sin^2(y-x)-\sin^4 z-\sin^2(y+x)$$
$$=2\sin^2 x\sin^2 y+2\sin^2 x\sin^2 z+2\sin^2 y\sin^2 z-4\sin^2 x\sin^2 y\sin^2 z-\sin^4 x$$
$$-\sin^4 y-\sin^4 z$$
$$=2\Sigma\sin^2 \Pi(xz)-4\Pi\sin^2 x-\Sigma\sin^4 x.$$

$$1-\sin(\Sigma)\sin\Pi(y+z-x)=1+\Sigma\sin^4 x+4\Pi\sin^2 x-2\Sigma\sin^2 \Pi(xy)....(1).$$

$$\Sigma\cos^4 x-2\Pi\cos^2 x+2\Pi\sin^2 x=\cos^4 x+\cos^4 y+\cos^4 z-2\cos^2 x\cos^2 y\cos^2 z$$
$$+2\sin^2 x\sin^2 y\sin^2 z=1+\sin^2 x+\sin^4 y+\sin^4 z+4\sin^2 x\sin^2 y\sin^2 z$$
$$-2\sin^2 x\sin^2 y-2\sin^2 x\sin^2 z-2\sin^2 y\sin^2 z$$
$$=1+\Sigma\sin^4 x+4\Pi\sin^2 x-2\Sigma\sin^2 \Pi(xy)....(2).$$

$\therefore$ (1)=(2). $\sin(a+b)\sin(a-b)=\sin^2 a-\sin^2 b$ gives (1).

No solutions of Problems 121, 122, 123, and 127 have been received. ED.

NOTES.

BIOGRAPHICAL SKETCH OF THE LATE HON. JOSIAH H. DRUMMOND.

The story of the life of Dr. Drummond when fully written would comprise a large part of the political history of the State of Maine during the last half a century. Only the leading facts in his life can here be narrated. For a more extended narrative of his life, the reader is referred to the newspapers of Portland, Maine, all of which at the time of his death gave very fully the leading events of his life.

Josiah H. Drummond was born in Winslow, Maine, August 30, 1827, and died at Portland, Maine, October 25, 1902. He graduated from Colby in 1846; read law in the office of Boutelle & Noyes in Waterville; was admitted to the bar of Maine in 1850, and to the bar of California, to which State he made a business trip, in 1851; returned to Maine and began the practice of law at Waterville; left the democratic party in 1855 on the anti-slavery issue and became one of the founders of the Republican party; was elected to the House of Representatives of Maine in 1857 and 1858, serving as Speaker; elected to State Senate in 1859, he was almost immediately after the beginning of the session elected attorney general and was three times re-elected; removed to Portland in 1860, was elected to the House from Portland in 1869, and declined a re-election in 1870; reeeived the degree of LL. D. from Colby College in 1871. He was a practicing lawyer, vice president of the trustees of Colby; a director of the Union Mutual Life Insnrance Company, and of the Union Safe Deposit and Trust Company, and clerk of the Maine Central corporation.

In 1875, Mr. Drummond was urged by his friends to become a candidate for United States Senator. With reluctance he allowed his name to be used, but was defeated owing largely to the influence of the late Thomas B. Reed. Mr. Reed and Mr. Drummond were strong personal friends, but he never asked "Tom" as he called him to the last, to change his position.

In 1875 and 1884 he was a delegate to the Republican National Convention, and in 1884 was the recognized Blaine leader.

While Mr. Drummond was a very busy man and actively interested in everything relating to the political interests of the State, and never allowing outside matters to interfere with his professional duties, he still found time to read some mathematical journals and solve some problems.

Dr. Drummond was very skillful in solving in a very elementary way some of the difficult problems in Diophantine Analysis. In this department of analysis he was especially interested, and the MONTHLY contains many of his excellent solutions.

For a portrait of Dr. Drummond see Vol. IV, No. 10. B. F. F.

This number completes the ninth volume of the MONTHLY. With Dr. Dickson as associate editor, we can assure our readers, with much confidence, that the tenth volume will contain much material of permanent value. In order that no hindrance be put in the way of future improvement, it is desirable, and necessary, that every one of our subscribers not only continue his subscription for the coming year, but also secure, if possible, one new subscriber. In order to encourage our subscribers to help increase our subscribtion list, we will send the MONTHLY one year to one old subscriber and one new one for $3.00. May we have the help of all in this matter? B. F. F.

By an order of the French Minister of War the use of logarithmic tables and calculations, based upon the *centesimal division of the right angle*, will be compulsory at the Polytechnic of Paris and at the military academy of Saint-Cyr after the year 1905. Five-place tables for the new and old system are issued by the French geographic service of the army. The experiences with the new division made by the French navy were most satisfactory. ARNOLD EMCH.